Chapter 3
Integral method for solving two-dimensional laminar boundary-layer problems
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Main Topics
• Idea of Integral method
• Karman momentum integral equation
• Pohlhausen’s method
• Thwaites’ method

Reading assignments:
As a counterpart of the similarity technique in the previous chapter, the integral method introduced in this chapter is also intended to reduce the PDE momentum equations to ODE equations, which can be solved easier. Being somewhat different from the previous technique, the integral method can be applied to all the boundary layer flows whether possess similarity nature or not. On the other hand, the results obtained by the integral method are concerned with the streamwise growth of the integral parameters of boundary layer, for instance, the boundary-layer momentum thickness $\theta$ and displacement thickness $\delta^*$. Detailed features of the velocity profiles are not obtainable.

The boundary layer momentum integral equations
(Karman integral relation after von Kaman)

Starting with the two-dimensional boundary layer equations (steady, incompressible)

**Momentum equation**

$$u \frac{\partial u}{\partial x} + \nu \frac{\partial u}{\partial y} = \frac{\partial p}{\partial x} + \frac{1}{\rho} \frac{\partial \rho}{\partial y}; \quad y = \mu \frac{\partial u}{\partial y}$$

**Continuity equation**

$$\frac{\partial u}{\partial x} + \frac{\partial \nu}{\partial y} = 0$$

Continuity $\partial y \times u$:

$$u \frac{\partial u}{\partial x} + u \frac{\partial \nu}{\partial y} = 0$$

$$\Rightarrow \frac{1}{2} \frac{\partial u^2}{\partial y} + \frac{\partial u \nu}{\partial y} - \nu \frac{\partial u}{\partial y} = 0$$
Wall shear stress

\[ \rho \frac{\partial \tau_{xw}}{\partial x} = \frac{\partial}{\partial x} (U \rho \frac{\partial U}{\partial x}) = \frac{\partial}{\partial x} (U \rho (\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} \frac{\partial U}{\partial y})) \]

Therefore, \( \int_0^x (\frac{\partial U}{\partial x} - U \frac{\partial U}{\partial x}) \, dx = \frac{\partial}{\partial x} \int_0^x U \rho \frac{\partial U}{\partial x} \, dx \)
The integral gets contributions only for values of $y < \delta; \ h > \delta$.

Hence, the integral is independent of $h$.

Further, the integral can be written as

$$\frac{d}{dx} \left[ C_\infty \int_0^h \frac{u(u_\infty - u)}{C_\infty^2} dy \right] + C_\infty \int_0^h \frac{u-u_\infty}{C_\infty} dy \frac{U_\infty}{p}$$

because $u_\infty = u_\infty(x)$, not function of $y$.

Define displacement thickness

$$s^* = \int_0^h \left(1 - \frac{u}{u_\infty} \right) dy$$

momentum thickness

$$\theta = \int_0^h \frac{u}{C_\infty} \left(1 - \frac{u}{u_\infty} \right) dy$$

Therefore,

$$\frac{d}{dx} \left( U_\infty^2 \theta \right) + s^* \frac{dU_\infty}{dx} + \frac{T_\infty}{\rho} = \frac{T_\infty}{\rho}$$

$$\Rightarrow \theta \frac{d}{dx} U_\infty + U_\infty \frac{d\theta}{dx} + s^* \frac{dU_\infty}{dx} = \frac{T_\infty}{\rho}$$

$$\Rightarrow \frac{d\theta}{dx} = \frac{T_\infty}{\rho U_\infty} - \theta \frac{dU_\infty}{dx}$$

$$\frac{d\theta}{dx} + (2 + H) \frac{\theta}{U_\infty} \frac{dU_\infty}{dx} = \frac{T_\infty}{\rho U_\infty^2}$$

where $H = \frac{s^*}{\theta}$, called shape factor.

The momentum integral equation is an ODE containing the variables of $\theta, T_\infty, H$, and $U_\infty$.

Karman momentum integral equation

For flat plate $t_z = \rho \frac{d\theta}{dx}$.
Normally, $\delta(x) = \int_0^h (1 - \frac{U}{U_\infty}) \, dy$

$\theta = \int_0^h \frac{U}{U_\infty} \, (1 - \frac{U}{U_\infty}) \, dy$

However, $\delta(x)$ and $\theta$ are not known, either.

Kármán momentum integral equation

$$\frac{d^2 \theta}{dx^2} = \frac{T_w}{\rho U_\infty^2} \left( 2 + \frac{1}{\rho} \frac{d U_\infty}{dx} \right) \theta \frac{d U_\infty}{dx} \frac{d^2 U_\infty}{dx^2}$$

Normally, $\delta(x)$ is known

$\theta$ is of interest to know, $\theta = \theta(x)$

But, there are three unknowns in the equation! $\theta$, $T_w$, $H$.

This led to a number of approximation methods reported in the literature to solve the equation. In the following, Pohlhausen's method will be introduced first, followed by the method of Thwaites.
Pohlhausen’s method (I)

The Karman integral equation

\[ \frac{U_m}{\delta} \frac{d\theta}{dx} + (2 + H) \frac{\theta}{U_m} \frac{dU_m}{dx} \]

where \( \theta \) (skin friction coefficient): \( \frac{C_f}{\frac{1}{2} \rho U_m^2} \)

The three variables \( C_f, \theta, H \) are directly related to the velocity profiles \( U(x, y) \).

Let us approximate \( U(x, y) \) by a one-parameter family of profiles.

**Free stream velocity**

\[ U(x, y) = U_\infty(x) f(y^*, \rho \infty(x)) \]

where \( y^* = y/\delta \), \( \rho \) is a non-dimensional parameter (to account for pressure gradient).

Pohlhausen’s method (II)

\[ u = U_\infty f \quad y^* = y/\delta \quad \frac{dy^*}{dy} = 1/\delta \]

\[ \delta^* = \int_0^h (1 - \frac{y}{\delta}) dy = \delta \int_0^h (1 - f)dy^* \]

\[ \theta = \int_0^h \tau \frac{dy}{dy^*} = \delta \int_0^h (1 - f)dy^* \]

\[ u = U_\infty f \quad \tau = \mu \frac{du}{dy} \bigg|_{y=0} = U_\infty \mu \frac{df}{dy} \bigg|_{y=0} = \frac{U_\infty \mu df}{\delta dy^*} \bigg|_{y=0} \]

\[ C_f = \frac{C_f}{\frac{1}{2} \rho U_m^2} \]

\[ c_r = \frac{1}{\frac{1}{2} \rho U_m^2} \frac{df}{dy^*} \bigg|_{y=0} = \frac{2 \mu}{\rho U_m \delta} \frac{df}{dy^*} \bigg|_{y=0} \]

\[ f(y^*, \rho \infty(x)) \]
This idea was first proposed by Pohlhausen with a velocity profile in the form:

\[ u(x,y) = a_1(x) y^* + a_2(x) y^{*2} + a_3(x) y^{*3} + a_4(x) y^{*4} \]

The boundary conditions for the momentum equation (boundary layer equation):

\[ y = 0 : \quad U = 0 = v \implies \text{therefore no } y^* \text{ term in the velocity profile expression} \]

\[ \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \]

\[ \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \]

\[ y = \delta : \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial y} = 0 \ldots \]
Apply the boundary conditions to the assumed velocity profiles:

\[ y^* = 0 : u = 0 \Rightarrow f(y^* = 0) = 0 \]

\[ \frac{\partial u}{\partial y} = -\frac{1}{y} \frac{dU_\infty}{dx} \]

\[ \frac{\partial^2 u}{\partial y^2} = \frac{\partial \left( y \frac{dU_\infty}{dx} \right)}{\partial y} \frac{dU_\infty}{dx} = \frac{1}{y} \frac{dU_\infty}{dx} \]

\[ \Rightarrow \frac{\partial^2 f}{\partial y^2} = -\frac{\partial^2 U_\infty}{\partial y^2} \frac{dU_\infty}{dx} \Rightarrow 2 a_2 \frac{dU_\infty}{dx} = -\frac{\partial^2 U_\infty}{\partial y^2} \]

Therefore, \[ a_2 = \frac{1}{2} \frac{\partial^2 U_\infty}{\partial y^2} \frac{dU_\infty}{dx} U_\infty \]

Here, \[ \frac{\partial^2 U_\infty}{\partial y^2} = \Lambda \]

\[ a_1 = -\frac{1}{2} \Lambda U_\infty \]

Since \[ y^* = \delta, \ u = U_\infty \]

\[ \Rightarrow \frac{1}{U_\infty} \left[ a_1 - \frac{1}{2} \Lambda U_\infty + a_3 + 4a_4 \right] = 1 \] \[ \Rightarrow \frac{1}{U_\infty} \left[ a_1 - \Lambda U_\infty + 3a_3 + 4a_4 \right] = 0 \]

\[ y^* = \delta, \ \frac{\partial u}{\partial y} = 0 \]

\[ \Rightarrow \frac{1}{U_\infty} \left[ a_1 - \Lambda U_\infty + 3a_3 + 4a_4 \right] = 0 \]

\[ y^* = \delta, \ \frac{\partial^2 u}{\partial y^2} = 0 \]

\[ \Rightarrow \frac{1}{U_\infty} \left[ -\Lambda U_\infty + 6a_3 + 12a_4 \right] = 0 \]
Solving a system of equations

\[ a_1 = \frac{1}{6} \Delta U \alpha + 2 \Delta U \alpha \]
\[ a_1 = \frac{1}{2} \Lambda U \beta \]
\[ a_3 = \Lambda U \beta (\Delta U = \Lambda) / 2 \]
\[ a_4 = \frac{1}{2} \Delta U \alpha \]

\[ f = \frac{U}{U_\infty} = \frac{1}{U_\infty} \left[ a_i y^i + a_2 y^2 + a_3 y^3 + a_4 y^4 \right] \]

Therefore,
\[ f = \frac{U}{U_\infty} = \frac{1}{U_\infty} \left[ \left( \frac{1}{2} \Delta U \alpha + 2 \Lambda U \beta \right) y^4 - \frac{1}{2} \Lambda U \beta \right] + \left( \frac{1}{2} \Lambda U \beta - 2 \Lambda U \beta \right) y^3 + (U_\infty - \frac{1}{2} \Lambda \beta) y^2 \]

\[ f = (2y^3 - 2y^2 + y^1) + \frac{\Lambda}{6} y^1(1 - y^1)^3 \]

The second term represents the effect associated with the pressure gradient.

\[ \Lambda = \frac{\varepsilon^2}{y} \frac{dU_\alpha}{dx} \]

\[ \frac{\bar{u}}{U_\infty} = f = (2y^3 - 2y^2 + y^1) + \frac{\Lambda}{6} y^1(1 - y^1)^3 \]

\[ -12 < \Lambda < 12 \]

\[ \frac{df}{dy} = (2 - 6y^2 + 4y^3) + \frac{\Lambda}{6} [(1 - y^1)^2 (1 - 3y^1)] \]

\[ \frac{df}{dy} \bigg|_{y=0} = 2 + \frac{\Lambda}{6} \quad \text{Separation occurs when} \quad \frac{df}{dy} \bigg|_{y=0} = 2 + \frac{\Lambda}{6} = 0 \quad \Lambda = -12 \]

\[ \frac{d^2f}{dy^2} = (1 - y^1)(12y^1(\frac{\Lambda}{6} - 1) - \Lambda) \]

Inflection point occurs if
\[ \frac{d^2f}{dy^2} = 0 \quad \Rightarrow 12y^1(\frac{\Lambda}{6} - 1) - \Lambda = 0 \]

Inflection point located at
\[ y^1 = \frac{\Lambda}{12(\Lambda - 1)} = \frac{\Lambda}{2(\Lambda - 6)} \]

\[ \Lambda > 12 \quad \text{inflection point} \quad \Rightarrow \frac{df}{dy} = 0 \quad \text{happens at} \quad y^1 < \frac{\Lambda}{2(\Lambda - 6)} \]

It is not correct physically if
inflection point located within boundary layer

\[ y^1 < \frac{\Lambda}{2(\Lambda - 6)} \]

\[ \Lambda > 12 \quad \text{i.e.} \quad u/U_\infty > 1 \quad \text{occurs within boundary layer!} \]
Refer to White (1974) (p.312, Fig. 4-22) for the velocity profiles with different $\Lambda$.

$f = f(x^*, \Lambda)$

$\Lambda > 12$, inflection point ($d^2f/dx^2 = 0$) happens at $y^* < 1$

$\Lambda = 12$, flow separation $dy/dx \rightarrow \infty$ \quad $\Lambda = -12$

$\Lambda = 12$ the profile is meaningful with the strongest

For the profiles with $\Lambda > 12$ or $\Lambda < -12$ they are physically irrelevant.

Calculate the boundary layer thickness over a flat plate by using Pohlhausen velocity

Pohlhausen velocity profile: $u/U_\infty = f = (2y^* - 2y^{**} - y^*)^2 + \frac{\Lambda}{6}y^*(1-y)^*$

For flat plate BL $\frac{\delta^*}{\delta} = \frac{\int_0^\infty (1-f) dy^*}{\int_0^\infty f(1-f) dy^*} = \frac{3}{10} \frac{\Lambda}{120} = 0.3$

$\frac{\delta}{\delta} = \int_0^\infty f(1-f) dy^* = \frac{37}{315} \frac{\Lambda}{945} \frac{\Lambda^2}{9072} = 0.37$

$\tau_w = \frac{\mu(\partial u/\partial y)_{wall}}{\rho U^2} = \frac{\mu(\partial u/\partial y)_{wall} / (\partial y / \partial \delta)_{wall}}{\rho U^2 \delta} = \frac{\mu}{\rho U^2} \frac{(2+H)U_\infty}{\rho U^2 \delta} \frac{\tau_w}{\rho U^2} = \frac{d\theta}{dx} U_\infty dU_\infty = \frac{\tau_w}{\rho U^2}$

Karman momentum integral equation

$\frac{d\theta}{dx} + (2+H) \frac{\theta}{U_\infty} \frac{dU_\infty}{dx} = \frac{\tau_w}{\rho U^2}$
Boundary thickness $\delta(x)$ of Pohlhausen velocity profile

$f = (2\gamma^* - 2\gamma^* - y^*) + \frac{\Lambda}{6} y^*(1 - y^*) = F(y^*) + \Lambda G(y^*)$

Substitute the velocity profile into the previous expressions ($\delta^*, \theta, H$) of Karman momentum integral equation:

$$\frac{d\delta}{dx} = \int_0^1 (1-f)dy^* = \frac{\Lambda}{10} \frac{\Lambda}{120}$$

$$\frac{\theta}{\rho U_x^2} = \int_0^1 f(1-f)dy^* = \frac{37}{315} \frac{\Lambda^2}{945} \frac{\Lambda}{9072}$$

$$\frac{\tau_x}{\rho U_x^2} = \frac{\mu(\hat{u}/\hat{y})_{y=0}}{\rho U_x^2} = \frac{1}{\rho U_x^2} \left[ \hat{u}(u/\hat{y})_y \hat{y} \right]_{y=0} = \frac{\mu}{\rho U_x^2} \left( \frac{2 + \Lambda}{6} \right)$$

Karman momentum integral equation:

$$\frac{d\theta}{dx} + (2 + H) \frac{\theta}{U_x} \frac{dU_x}{dx} = \frac{\tau_x}{\rho U_x^2}$$

$$\frac{d\delta}{dx} = f(\delta, \Lambda, \Lambda, \frac{\Delta}{U_x} \frac{dU_x}{dx})$$

How to find the boundary layer thickness $\delta(x)$ or $\Lambda$ in BL with pressure gradient by using Pohlhausen method?
Substitute the velocity profiles into the previous expressions of the momentum integral equation can be expressed as:

\[
\frac{dS}{dx} = f\left(\theta, \frac{d\theta}{dx}, \frac{d^2\theta}{dx^2}, \frac{d\theta}{dx} \cdot \frac{d^2\theta}{dx^2}\right)
\]

One variable due to \(\frac{d\theta}{dx} = \frac{d\theta}{dx} \cdot \frac{d^2\theta}{dx^2}\) results in an equation of boundary layer thickness, the equation is re-grouped for solving:

\[
\lambda = \frac{\theta^2}{\frac{d\theta}{dx}} \frac{dU_\infty}{dx}
\]

See White (1974) p. 312 for the improvement of the method. Instead of an equation of boundary layer thickness, the equation is re-grouped for solving:

\[
\lambda = \frac{\theta^2}{\frac{d\theta}{dx}} \frac{dU_\infty}{dx}
\]

\[
\text{Thwaites' method}
\]

To eliminate the disadvantage due to one choose \(\theta\), instead of \(\frac{d\theta}{dx}\), as the new primary variable

\[
\lambda = \frac{\theta^2}{\frac{d\theta}{dx}} \frac{dU_\infty}{dx}
\]

Multiply the momentum integral equation by \(\frac{d\theta}{dx}\):

\[
\left(\frac{d\theta}{dx} = \frac{U_\infty \theta}{\rho U_\infty^2} - (H + \tau) \frac{dU_\infty}{dx} \right) \frac{d\theta}{dx}
\]

\[
\Rightarrow \frac{\theta}{\rho U_\infty} = \frac{U_\infty \theta}{\frac{d\theta}{dx}} + \frac{\theta^2}{\frac{d\theta}{dx}} \frac{dU_\infty}{dx} (H + \tau)
\]

This is an equation of \(\theta\)!
Boundary thickness of Pohlhausen velocity profile

\[ f = (2y^* - 2y^{*3} - y^{*4}) + \frac{\Lambda}{6}y^*(1 - y^*)^3 = F(y^*) + AG(y^*) \]

Substitute the velocity profiles into the previous expressions of \( \delta^*, \theta, H \)

\[
\frac{\delta^*}{\delta} = \frac{\int_0^\delta (1 - f) dy}{\Lambda} = \frac{3}{10} - \frac{\Lambda}{120} \quad \Lambda = \frac{\delta^* U_\infty}{v} dx
\]

\[
\frac{\theta}{\theta} = \frac{\int_0^\theta f(1 - f) dy}{\Lambda} = \frac{37\Lambda}{315} - \frac{\Lambda^2}{945} - \frac{\Lambda^3}{9072} \quad H = \frac{\delta^*}{\theta} = \frac{3}{10} - \frac{\Lambda}{120}
\]

Use new primary variable \( \lambda \)

\[
\lambda = \frac{\theta^* dU_{\infty}}{v} = \frac{\theta^*}{\delta^*} \Lambda = \left(\frac{37\Lambda}{315} - \frac{\Lambda^2}{945} - \frac{\Lambda^3}{9072}\right) \Lambda \quad \Lambda = f(\lambda)
\]

Karman momentum integral equation in momentum thickness

\[
U_\infty \frac{dz}{dx} + (2 + H) \lambda = \frac{\tau_u \theta}{\mu U_\infty} \quad \lambda = \frac{\theta^*}{\delta^*} \Lambda
\]

\[
\frac{\tau_u \theta}{\mu U_{\infty}} = \frac{\mu}{\rho U_{\infty} \theta} (2 + \Lambda) \quad \frac{\tau_u \theta}{\mu U_{\infty}} = \frac{\theta^*}{\delta^*} \left(\frac{12 + \Lambda}{6}\right)
\]

\[
U_\infty \frac{dz}{dx} + (2 + H(\lambda)) \lambda = \frac{\tau_u \theta}{\mu U_{\infty}} \quad \lambda = \frac{z U_{\infty}}{dx}
\]

\[
U_\infty \frac{dz}{dx} + (2 + H(\lambda)) \lambda = S(\lambda)
\]

\[
dz{dx} = \left[\frac{2 S(\lambda) - 2\lambda H(\lambda) + 2]}{U_\infty}\right] = \frac{F(\lambda)}{U_\infty}
\]

\[
dz{dx} = \frac{F(\lambda)}{U_\infty} \quad \text{Solve this non-linear 1st ODE to find } z \quad \Longrightarrow \quad z = \frac{\theta^*}{\delta^*} \iff \theta
\]
Example: Calculate the boundary layer thickness of Pohlhausen velocity profile over a flat plate by using Thwaites’ method and compare the result with Blasius solution.

**Pohlhausen velocity profile:**

\[
\frac{U}{\nu} = f = (2y^2 - 2y^3y^4 + \frac{\Lambda}{6}y^6(1 - y^2))^3
\]

For flat plate BL:

\[
\frac{d^2}{\nu^2} \frac{dU}{dx} = 0 \quad \frac{\Lambda}{\nu^2} \frac{\partial}{\partial \nu} \left( \frac{\nu}{\nu} \right) = 0 \quad \frac{\Lambda}{\nu^2} \frac{\partial}{\partial \nu} \left( \frac{\nu}{\nu} \right) = 0
\]

\[
F(\lambda) = F(0) = 2\nu(\lambda) - 2\nu(\lambda) + 2 = 0.4698
\]

**Blasius solution**

\[
\frac{dz}{dx} = \frac{F(\lambda)}{U_\infty} \quad d\left( \frac{\lambda}{\nu} \right) \frac{dx}{U_\infty} = 0.4698 \quad \theta = \frac{0.4698}{\nu U_\infty} \quad \frac{1}{\nu U_\infty} = 0.685 \quad \frac{1}{\nu U_\infty} = 0.664
\]

\[
H = \frac{\delta}{\nu} = f_1(\lambda) = \frac{945}{370} \quad \delta = \frac{945}{370} \frac{\theta}{\nu} \quad \frac{945}{370} \frac{\theta}{\nu} = 0.685 \quad \frac{945}{370} \frac{\theta}{\nu} = 1.75
\]

\[
\tau_\nu = \frac{74}{315} \mu U_\infty \quad \tau_\mu = \frac{74}{315} \frac{\mu U_\infty}{\nu(1/2)} = 0.343 \quad \sqrt{Re_x} \quad \frac{\nu U_\infty}{\sqrt{Re_x}}
\]

\[
S(\lambda) = \frac{74}{315} \quad \frac{\nu U_\infty}{\sqrt{Re_x}} = 0.4698
\]

**Pohlhausen Boundary Layer Velocity Profile**
Example: Calculate the momentum thickness and skin friction of boundary layer with pressure gradient by using Thwaites’ method

\[ \tau_{w} = \frac{\mu U}{\theta} S(\lambda) \]

Shear correlation

\[ H = \frac{\delta^{*}}{\delta} \approx H(\lambda) \]

Shake-factor correlation

Consider \( \frac{U_{w}}{\nu} \frac{\theta^{*}}{dx} = \frac{1}{2} \frac{U_{w}}{\nu} \frac{d\theta^{*}}{dx} \)

Therefore, the equation becomes

\[ \frac{U_{w}}{\nu} \frac{d\theta^{*}}{dx} = 2[S(\lambda) - \lambda(H + 2)] = F(\lambda) \]

Pohlhausen Method

\[ \Lambda = \frac{\delta^{*}}{\theta} \frac{dU_{w}}{dx} \quad S(\lambda) = \frac{\tau_{w}}{\mu U_{w}} \left( \frac{37}{315} \frac{\Lambda}{945} \right)^{12} + \frac{\Lambda}{9072} \]

\[ \lambda = \frac{\delta^{*}}{\theta} \frac{dU_{w}}{dx} \frac{\delta^{*}}{dx} \quad H(\Lambda) = \frac{\delta^{*}}{\delta} = \frac{3}{10} \frac{\Lambda}{120} \]

\[ \lambda = \left( \frac{37}{315} \frac{\Lambda}{945} \right)^{12} + \frac{\Lambda}{9072} \]

Thwaites (1949)

\[ S(\lambda) = (\lambda + 0.09)^{d_{12}} \quad H(\Lambda) = 2.0 + 4.41z = 83.52 \quad 854z^{-1} = 333z^{4} + 4576z^{5} \]

\[ z = 0.25 - \lambda \]

Flow separation occurs at \( S(\lambda) = 0 \)

\[ \lambda_{sep} \approx -0.09 \]

White, F. M., Viscous fluid flow. McGraw-Hill, 2nd edition, Chapter 4.6.6
Cebeci and Bradshaw (1977) empirical relationship $S(\lambda)$ & $H(\lambda)$

For $0 < \lambda < 0.1$, $\frac{d\lambda}{d\varepsilon} < 0$ (favourable pressure gradient)

\[ S(\lambda) = 0.22 + 1.57\lambda - 1.8\lambda^2 \]

For $-0.1 < \lambda < 0$, $\frac{d\lambda}{d\varepsilon} > 0$ (adverse pressure gradient)

\[ S(\lambda) = 0.22 + 1.402\lambda + \frac{0.018\lambda}{\lambda + 0.107} \]

\[ H(\lambda) = \frac{0.0731}{0.14 + \lambda} + 2.088 \]

\[ F(\lambda) = 2[S - \lambda(H + 2)] \]

\[ F(\lambda) = 2\left[ \frac{37}{315} - \frac{\lambda}{945} \right] - \lambda \left( \frac{3}{10} - \frac{\lambda}{9072} \right) \]

\[ F(\lambda) = 2\left[ \frac{37}{315} - \frac{\lambda}{945} \right] - \lambda \left( \frac{3}{10} - \frac{\lambda}{9072} \right) \]

\[ F(\lambda) = 2\left[ \frac{37}{315} - \frac{\lambda}{945} \right] - \lambda \left( \frac{3}{10} - \frac{\lambda}{9072} \right) \]

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\[ F(\lambda) = 2\left[ \frac{37}{315} - \frac{\lambda}{945} \right] - \lambda \left( \frac{3}{10} - \frac{\lambda}{9072} \right) \]

Empirical correlation reduced from the data of a number of studies. (white, Fig. 4-22)

\[ F(\lambda) \approx 0.45 - 6\lambda \quad \text{for} \quad 0.1 < \lambda < 0.1 \]

Thwaites (1949)

\[ F(x) = 0.45 - 6.0x \]

Check the relation

\[ \frac{U_x}{v} \frac{d^2 \theta}{dx} = F(\lambda) = 0.45 - 6\left(\frac{\theta^2}{v} \frac{dU_x}{dx}\right) \]

\[ 6\frac{\theta^2}{v} \frac{dU_x}{dx} + \frac{U_x}{v} \frac{d\theta^2}{dx} = 0.45 \]

Multiply by \( U_x^5 \) (the whole equation)

\[ \frac{\theta^2}{v} \frac{dU_x^6}{dx} + \frac{U_x^6}{v} \frac{d\theta^2}{dx} = 0.45U_x^5 \Rightarrow \frac{1}{v} \frac{d(\theta^2U_x^6)}{dx} = 0.45U_x^5 \]

\[ \frac{\theta(x)^2U_x^6(x)}{v} = 0.45 \int_0^x U_x^5 dx + \frac{\theta(0)^2U_x^6(0)}{v} \]

\( U_x(x) \) known \( \Rightarrow \theta \) calculated.
Step 1: Potential flow $U(x)$

Step 2: find $\theta(x)$

$$\frac{\theta(x)^2U^6(x)}{\nu} = 0.45 \int_0^x U' \, dx + \frac{\theta(0)^2U^6(0)}{\nu} \Rightarrow \theta(x)$$

Step 3: find $\lambda(x)$

$$\lambda = \frac{\theta^2}{\nu} \frac{dU}{dx} \Rightarrow \lambda(x)$$

Step 4: use empirical relationship $S(\lambda)$ & $H(\lambda)$

Step 5: find skin friction from $S(\lambda)$

$$\tau_w = \frac{\mu U_w}{\theta} S(\lambda)$$

Step 6: find displacement thickness $\delta^*$ from $H(\lambda)$

$$\delta^* = \theta H(\lambda)$$

4-6.7 Application to the Howarth Decelerating Flow

Howarth’s linear decelerating flows $U_\infty(x) = U_\infty(1 - \frac{x}{L})$

To illustrate the simplicity and accuracy of Thwaites’ method, we apply it to the Howarth linearly decelerating flow of Eq. (4-114) and Fig. 4-20, with $\frac{dU}{dx} = -\frac{U_0}{L} = \text{const}$, the momentum thickness is computed, approximately, from Eq. (4-138):

$$\theta^2 = \frac{0.45\nu}{U_0^2(1 - x/L)^5} \int_0^x U_0^6 \left(1 - \frac{x}{L}\right)^5 \, dx = 0.075 \frac{\mu U_0 L}{U_0^2} \left[ \left(1 - \frac{x}{L}\right)^{-5} - 1 \right]$$

from which, by definition,

$$\lambda = \frac{\theta^2}{\nu} \frac{dU}{dx} = -0.075 \left[ \left(1 - \frac{x}{L}\right)^{-5} - 1 \right]$$

(4-142)

With $\lambda(x)$ given by this (approximate) expression, we could then compute the wall shear $\tau_w(x)$ from the function $S(\lambda)$ in Table 4-4 or Fig. 4-23b. Let us save that until Sec. 4-7 as a comparison.

Given $\lambda(x)$, the separation point is predicted by

$$\lambda_{\text{sep}} \approx -0.09$$

or

$$\frac{x_{\text{sep}}}{L} = 1 - (2.2)^{-1/5} = 0.123$$
HW: momentum thickness of flow over a circular cylinder

Consider boundary layer on the front surface of a circular cylinder

\[ U_\infty = 2 \frac{U_0 \sin \phi}{\sin 2\phi}, \quad \phi = \frac{x}{R_0} \]

Find \( \theta, C_f \) and \( H \) at \( \phi = 40^\circ \)

\[ \frac{\theta^3 U_0^6}{\nu} = 0.45 \int_0^L x U_\infty^5 \, dx + \left( \frac{\theta^3 U_0^6}{\nu} \right)_{x=0} \]

Non-dimensionalizing the equation by:

\[ x^* = \frac{x}{L}, \quad U^* = \frac{U}{U_\infty}, \quad R_e = \frac{U_\infty L}{\nu}, \quad L = R_0 \]

\[ \left( \frac{\theta}{R_0} \right)^2 \left( \frac{U_\infty}{2} \right)^5 \int_0^L x^* U^* dx^* = 0.45 \left( \frac{\theta}{R_0} \right)^2 \left( \frac{U_\infty}{2} \right)^5 \int_0^L x^* dx^* + \left( \frac{\theta^3 U_0^6}{\nu} \right)_{x=0} \]

Here \( U^* = \frac{U_\infty}{U_0} = 2 \sin \phi \), \( x^* = x^* \)

\[ \Rightarrow \frac{\theta}{R_0} \left( \frac{R_0}{R_0} \right)^{\frac{1}{2}} = \left( \frac{0.45}{(2 \sin 45^\circ)^6} \int_0^{2 \sin 45^\circ} (2 \sin \phi)^5 \, d\phi \right) \]

\[ = 0.213 \]
(with $\theta$ obtained from the above expression,

$$\lambda = \frac{\theta^2}{\nu} dU_\infty $$

Then, find $H(x)$ \rightarrow find $S^*$

find $S\theta(x)$ \rightarrow find $T\theta$)

4-6.8 Application to Laminar Flow Past a Circular Cylinder

Both the accuracy and the dilemma of a bluff-body boundary-layer calculation are illustrated by the circular cylinder. In terms of the dimensionless arc length $x^* = x/a$, where $a$ is the cylinder radius, the potential-flow velocity distribution is

$$\frac{U}{U_\infty} = 2\sin x^* = 2.0x^* - 0.333x^3 + 0.0167x^5 \ldots \quad (4-143)$$

from which we can easily generate boundary-layer solutions of any type (integral, series, or digital computer). Calculations of this type have been made. Separation is predicted at an angle $x^* = \phi = 104.5^\circ$ in numerical results by Terrill (1960), which we might (mistakenly) think would be reproduced in an experimental laminar cylinder flow.
Unfortunately, as discussed earlier in Sec. 1-2, the broad wake caused by bluff-body separation is a first-order effect, i.e., it is so different from potential flow (see Figs. 1-5 and 1-6) that it alters $U(x)$ greatly everywhere, even at the stagnation point. For example, the experiment of Hiemenz (1911) for a cylinder at a Reynolds number $Re_q = U_\infty a/\nu = 9500$ fit the polynomial

$$\frac{U}{U_\infty} \approx 1.814x^2 - 0.271x^3 - 0.0471x^5$$

(4-144)

which is quite different from potential flow. Even the stagnation velocity gradient