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混合式複合材料之熱弾性分析(2/3)
Thermoelastic Analysis for Hybrid Composites (2/3)

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中華民國 109 年 6 月 8 日
中文摘要

依原計畫規劃接續第一年之工作，本年度我們針對一般異向性彈性材料進行邊界元素
分析，推導其中所需之熱彈基本解(同時推導複數型態解與實數型態解)，並將與熱效應相
關之體積分轉換為邊界積分，進而將推導結果應用於多連通區域之問題如多孔洞、多裂縫
或多異質等。截至目前為止，所有預期之工作皆已順利達成。

關鍵詞：混合式複合材料，熱彈性分析，異向性彈性力學，邊界元素法

Abstract

According to our original plan and following the first year’s results of our project, this year
we focus on the boundary element formulation of two-dimensional anisotropic thermoelastic
analysis. Based upon the requirement of boundary element formulation, we derived the thermal
fundamental solutions, both in complex form and real form. Furthermore, to achieve a real
formulation of boundary integral, we reduce the associated thermal body integral to a surface
integral and apply all the derivation to the problems with multi-connected domains such as the
problems with multiple holes, cracks, or inclusions. Till now, all these planned works have been
completed.

Keywords: hybrid composites, thermoelastic analysis, anisotropic elasticity, boundary element
method

1. BEM for 2D anisotropic magneto-electro-elastic thermal analysis

When an anisotropic magneto-electro-elastic (MEE) material under thermal loading is
considered, the basic equations such as the constitutive laws, the heat conduction, energy
equation, strain-displacement relation, equilibrium equations, electric balance, and Gauss
equation for the uncoupled steady state thermal MEE analysis can be expressed as (Nowacki,
1962)

\[
\begin{align*}
\sigma_{ij} &= C^{E,H}_{ijkl} \varepsilon_{kl} - q^{E}_{ijkl} H_k - q^{H}_{ijkl} E_k - \beta_{ij} T \\
D_j &= e^{H}_{ijkl} \varepsilon_{kl} + \omega^{E,H}_{ijkl} E_k + m^{E,H}_{ijkl} H_k + \beta_{ij} T \\
B_j &= q^{E}_{ijkl} \varepsilon_{kl} + m^{E}_{ijkl} E_k + q^{E,H}_{ijkl} H_k + \beta_{ij} T \\
h_i &= -k_{ij} T_{,j}, \quad h_{i,j} = 0, \quad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \\
\sigma_{ij,l} &= 0, \quad D_{i,j} = 0, \quad B_{j,i} = 0, \quad i, j, k, l = 1, 2, 3.
\end{align*}
\]

In the above, \(\sigma_{ij}, \varepsilon_{ij}, D_j, E_j, B_j,\) and \(H_j\) denote, respectively, the stress, strain, electric
displacement, electric field, magnetic induction, and magnetic field; \(C^{E,H}_{ijkl}, \varepsilon_{ijkl}, q^{E}_{ijkl}, \omega^{E,H}_{ijkl}, m^{E,H}_{ijkl},\)
and \(q^{E,E}_{ijkl}\) are, respectively, the elastic stiffness tensor at constant electric and magnetic field,
the piezoelectric stress tensor at constant magnetic field, the piezomagnetic stress tensor at
constant electric field, the dielectric permittivity tensor at constant strain and magnetic field, the
magneto-electric coefficients at constant strain, and the magnetic permeability tensor at constant
strain and electric field; \(k_{ij}, \beta_{ij}\) are the heat conduction coefficients and thermal moduli,
respectively; \(u_i, T\) and \(h_i\) are, respectively, the displacement, temperature and heat flux; the
subscript comma stands for differentiation with respect to the coordinate variable \(x_i, i = 1,2,3,\).
If body forces are omitted, the boundary integral equations for anisotropic magneto-electro-thermoelasticity can be written as (Hwu, et al., 2019)

\[
c_{ij}(\xi)u_{ij}(\xi) + \int_{\Gamma} t_{ij}^*(\xi, x)u_{ij}(x)d\Gamma(x) = \int_{\Gamma} u_{ij}^*(\xi, x)t_j(x)d\Gamma(x) + \int_{\Omega} \beta_{jk}u_{ij,k}(\xi, x)T(x)d\Omega(x), \quad i, j, k = 1, 2, 3, 4, 5.
\]

In the above, \(\xi = (\xi_1, \xi_2)\) and \(x = (x_1, x_2)\) represent, respectively, the source point and field point of the boundary integral equations. \(\Gamma\) and \(\Omega\) are the boundary and body of the MEE solid. The symbol \(\int_{\Gamma}\) denotes an integral taken in the sense of Cauchy principal value. \(c_{ij}(\xi)\) are the free term coefficients dependent on the location of the source point \(\xi\), which equals to \(\delta_{ij}/2\) for a smooth boundary and \(c_{ij} = \delta_{ij}\) for an internal point. The symbol \(\delta_{ij}\) is the Kronecker delta, i.e., \(\delta_{ij} = 1\) when \(i = j\) and \(\delta_{ij} = 0\) when \(i \neq j\). In practical applications, \(c_{ij}(\xi)\) can be computed by considering rigid body motion. \(u_{ij}(x)\) and \(t_j(x)\) are the generalized displacements and surface tractions at the field point \(x\). \(u_{ij}^*(\xi, x)\) and \(t_j^*(\xi, x)\) are, respectively, the fundamental solutions of generalized displacements and tractions.

Equation (2) shows that the difference between elastic and MEE materials is the sub-indices which are expanded from 3 to 5 for MEE analysis, and the fourth and fifth components of generalized displacements and tractions are related to the electric displacement \(D_i\), electric field \(E_i\), magnetic induction \(B_i\), and magnetic field \(H_i\), as those shown below

\[
-E_i = u_{ik,j} = 2\varepsilon_{kj}, \quad -H_i = u_{k,j} = 2\varepsilon_{kj}, \quad D_j = \sigma_{kj}, \quad B_j = \sigma_{kj}, \quad j = 1, 2, 3.
\]

Moreover, the difference between isothermal and thermal elastic analysis is the second term at the right hand side of (2), which is a body integral reflecting the thermal effects through temperature change \(T(x)\) and thermal moduli \(\beta_{ijk}\). Because this term is a body integral instead of a boundary integral, without considering the influence of heat source to avoid body discretization on BEM an exact transformation for this body integral has been made as (Shiah, et al., 2014)

\[
\int_{\Omega} \beta_{jk}u_{ij,k}^*(\xi, x)T(x)d\Omega(x) = -\int_{\Gamma} k_{jk}[v^*_j(\xi, x)T^*_{jk}(x) - v^*_{ij,k}(\xi, x)T(x)]n_j(x)d\Gamma(x),
\]

where

\[
k_{jk}v^*_{ij,k}(\xi, x) = \beta_{jk}u_{ij,k}^*(\xi, x),
\]

and \(n_j(x)\) is the normal direction at the field point \(x\).

If the influence of heat source is considered and the heat generation rate \(\gamma\), which satisfies the relation that \(h_{ij} = \gamma\), is assumed to be constant within the domain, equation (4) should be modified as (Shiah, et al., 2018)

\[
\int_{\Omega} \beta_{jk}u_{ij,k}^*(\xi, x)T(x)d\Omega(x) = -\int_{\Gamma} \{\gamma w^*_j(\xi, x) + k_{jk}[v^*_j(\xi, x)T^*_{jk}(x) - v^*_{ij,k}(\xi, x)T(x)]\}n_j(x)d\Gamma(x),
\]

where

\[
w^*_j(\xi, x) = v^*_j(\xi, x).
\]

If the uncoupled steady state magneto-electro-thermoelastic problem is considered, as shown in the basic equations, the heat flux \(h_i\) and the temperature \(T\) can be determined independently from the relations \(h_i = -k_{ij}T_{ij}\) and \(h_{ij} = 0\) as well as the thermal boundary conditions for the
problem. Therefore, the temperature $T(x)$ in (4a) or (5a) is treated as a known function, which can be determined from any other method such as finite element method or boundary element method. Thus, to deal with the boundary integral obtained in (4) or (5) for thermoelastic analysis, the only function remained to be determined is the thermal fundamental solution $v_i^*$.  

2. Thermal fundamental solution

From the equality of (4b), we know that the thermal fundamental solution $v_i^*$ is related to the fundamental solution of displacements $u_i^*$, which can be expressed by the matrix form of Stroh formalism as (Hwu, 2010)

$$[u_i^*] = \frac{1}{\pi} \text{Im}\{A < \text{ln}(z_a - \hat{z}_a) > A^T\},$$

$$z_a = x_1 + \mu_a x_2, \quad a = 1, 2, 3, 4, 5,$$

where $\text{Im}$ denotes the imaginary part of a complex number; the angular bracket $<>$ stands for the diagonal matrix in which each component is varied according to the subscript $\alpha$, e.g.,

$$\langle g(z) \rangle = \text{diag}[g(z_1), g(z_2), g(z_3), g(z_4), g(z_5)];$$

$\mu_\alpha$ and $a_\alpha$, $k = 1, 2, 3, 4, 5$, are, respectively, the material eigenvalues and material eigenvectors.

Substituting (6) into the equality (4b), the thermal fundamental solution $v_i^*$ has been derived and expressed in complex matrix form as (Shiah, et al., 2014)

$$v^* = \frac{1}{\pi} \text{Im}\{A < k_\alpha^{-1}(z_a - \hat{z}_a)[\text{ln}(z_a - \hat{z}_a) - 1] > \beta_{\alpha}\},$$

where

$$k_\alpha = k_{11} + 2k_{12} \mu_a + k_{22} \mu_a^2, \quad \beta_{\alpha} = A^T \beta_{1} + < \mu_a > A^T \beta_{2},$$

and $\beta_1$ and $\beta_2$ are defined as

$$\beta_1 = \begin{bmatrix} \beta_{11} \\ \beta_{21} \\ \beta_{41} \\ \beta_{51} \end{bmatrix}, \quad \beta_2 = \begin{bmatrix} \beta_{12} \\ \beta_{22} \\ \beta_{42} \\ \beta_{52} \end{bmatrix},$$

for the present case of MEE materials.

Note that in the derivation of (Shiah, et al., 2014) only the anisotropic elastic materials are considered. With the benefit of Stroh formalism, same solution form can also be applied to the cases of MEE materials. The only difference is that all the related vectors (or matrices) are now expanded from $3 \times 1$ (or $3 \times 3$) to $5 \times 1$ (or $5 \times 5$). Due to the multi-valued characteristics of complex logarithmic function, it has been proved by Shiah, et al. (2018) that an extra line integral crossing the domain surface is required to evaluate (4a) for the problem with multi-connected domain. Therefore, even only the boundary integral is involved in the right hand side of (4a) we still need to provide the temperature information inside the domain, which will reduce the value of BEM.

**Derivation of the real form thermal fundamental solution**

To avoid the problem induced by the complex logarithmic function, instead of the complex form fundamental solution of displacement as shown in (6) we consider the real form fundamental solution converted by using the Stroh’s identities as (Hwu, 2010)
\[ [u_y] = U^* = -\frac{1}{2\pi} \{(\ln r)H + \pi[\tilde{N}_1(\theta)H + \tilde{N}_2(\theta)S^T]\}, \]  
(10a)

where

\[ \tilde{N}_i(\theta) = \frac{1}{\pi} \int_0^\theta N_i(\omega)d\omega, \quad i = 1, 2. \]  
(10b)

In (10), \((r, \theta)\) is the polar coordinate with the origin located at the source point \(\hat{x} = (\hat{x}_1, \hat{x}_2)\); \(S\) and \(H\) are two real matrices usually called Barnett-Lothe tensors, which are related to \(\tilde{N}_1(\theta)\) by \(S = \tilde{N}_1(\pi)\) and \(H = \tilde{N}_2(\pi)\); and \(N_i(\omega), \ i = 1, 2, 3\), are the submatrices of the generalized fundamental elasticity matrix \(N(\omega)\) of Stroh formalism.

With the real form solution (10a), the right hand side of (4b) can be expressed in matrix form as

\[ [\beta_{jk} u_{y,k}] = U_{1}^* \beta_1 + U_{2}^* \beta_2, \]  
(11)

Substituting (10a) into (11) and using chain rule for the differentiation, we obtain

\[ U_{1}^* \beta_1 + U_{2}^* \beta_2 = \frac{1}{r} g(\theta), \]  
(12a)

where

\[ g(\theta) = -\frac{1}{2\pi} \{[H\beta_1^*(\theta) + [N_1(\theta)H + N_2(\theta)S^T] \beta_2^*(\theta)\}, \]  
(12b)

\[ \beta_1^*(\theta) = \cos \theta \beta_1 + \sin \theta \beta_2, \quad \beta_2^*(\theta) = -\sin \theta \beta_1 + \cos \theta \beta_2. \]

By referring to the complex form solution (8) and the result of (12a), to find \(v_i^*\) from (4b) we may let

\[ v_i^* = r \ln rv_i(\theta) + rv_x(\theta), \]  
(13)

where \(v_i(\theta)\) and \(v_x(\theta)\) are two unknown vectors to be determined. With (13) and the aid of chain rule differentiation, the left hand side of (4b) becomes

\[ [k_{jk} v_{y,k}] = \frac{1}{r} \left\{k_0 v_1 + 2k_{12}^*(\theta)v_1^* + k_{11}^*(\theta)(v_2 + v_x^*)\right\} + \frac{\ln r}{r} k_{11}^*(\theta)(v_1 + v_x^*), \]  
(14)

where the prime \(\cdot^*\) and double prime \(\cdot^{**}\) denote, respectively, the first and second derivatives with respect to the variable \(\theta\), and

\[ k_0 = k_{11} + k_{22}, \]

\[ k_{12}^*(\theta) = -k_{11} \sin \theta \cos \theta + k_{12} (\cos^2 \theta - \sin^2 \theta) + k_{22} \sin \theta \cos \theta, \]

\[ k_{11}^*(\theta) = k_{11} \sin^2 \theta - 2k_{12} \sin \theta \cos \theta + k_{22} \cos^2 \theta. \]

Equality between (14) and (12a) leads to

\[ v_1 + v_1^* = 0, \quad v_2 + v_2^* = h(\theta), \]  
(16a)

where

\[ h(\theta) = \frac{1}{k_{11}^*(\theta)} \left\{g(\theta) - k_0 v_1 - 2k_{12}^*(\theta)v_1^*\right\}. \]  
(16b)

From (16a), we have

\[ v_1 = \cos \theta v_x + \sin \theta v_s, \quad v_2 = \cos \theta v_x + \sin \theta v_s + v_x^{(p)}(\theta), \]  
(17)

where \(v_x, v_y, e_x,\) and \(e_y\) are four arbitrary constant vectors, and \(v_x^{(p)}(\theta)\) are the particular solutions related to \(h(\theta)\). Substituting \(v_1(\theta)\) obtained in (17) into \(h(\theta)\) of (16b), we have

\[ h(\theta) = g(\theta) - k_1^*(\theta)v_x - k_2^*(\theta)v_s, \]  
(18a)

where
To avoid getting the solution with the terms of $\theta \cos \theta$ and $\theta \sin \theta$ which will cause discontinuity, we choose a specific $v_c$ and $v_s$ to let $h(\theta)$ be expanded without the terms of $\sin \theta$ and $\cos \theta$. In other words, if $h(\theta)$ is expanded by Fourier series as

$$h(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta),$$

(19a)

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\theta) \, d\theta,$$

(19b)

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} h(\theta) \cos n\theta \, d\theta, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} h(\theta) \sin n\theta \, d\theta, \quad n = 1, 2, 3, \ldots, \infty,$$

its 2nd and 3rd terms associated with $\cos \theta$ and $\sin \theta$ should vanish. Under this consideration, substituting (18a) into (19b) with $a_i = b_i = 0$, we get

$$v_c = \frac{1}{\Delta} \{k_{ss} g_s - k_{sc} g_c\}, \quad v_s = \frac{1}{\Delta} \{k_{cc} g_c - k_{cs} g_s\},$$

(20a)

where

$$g_c = \int_{-\pi}^{\pi} g_0(\theta) \cos \theta d\theta, \quad g_s = \int_{-\pi}^{\pi} g_0(\theta) \sin \theta d\theta,$$

$$k_{cc} = \int_{-\pi}^{\pi} k_c(\theta) \cos \theta d\theta, \quad k_{sc} = \int_{-\pi}^{\pi} k_s(\theta) \cos \theta d\theta,$$

$$k_{cs} = \int_{-\pi}^{\pi} k_c(\theta) \sin \theta d\theta, \quad k_{ss} = \int_{-\pi}^{\pi} k_s(\theta) \sin \theta d\theta,$$

$$\Delta = k_{cc} k_{ss} - k_{cs} k_{sc}.$$

(20b)

Through the above derivation, we know that the solution obtained in (18a) with (20) will make (19a) becomes a series with $a_i = b_i = 0$. Thus, the solution obtained in (17)2 can be written as

$$v_2(\theta) = \cos \theta v_c + \sin \theta v_s + a_0 + \sum_{n=2}^{\infty} \frac{1}{1-n} (a_n \cos n\theta + b_n \sin n\theta).$$

(21)

To see whether the terms associated with the constant vectors $e_c$ and $e_s$ can be ignored, we consider the solution

$$v_0' = r(\cos \theta e_c + \sin \theta e_s).$$

(22)

With (22), it can be proved that $k_{jk} v_{0,jk} = 0$, which means that the existence of $e_c$ and $e_s$ will not influence the satisfaction (4b). In other words, the constant vectors $e_c$ and $e_s$ can be chosen arbitrarily. For the convenience of calculation, we let

$$e_c = e_s = 0.$$

(23)

Combining the results of (13), (17)1, (21), and (23), the real form thermal fundamental solution $v^*$ can finally be simplified as

$$v^* = r \ln r (\cos \theta v_c + \sin \theta v_s) + r \left\{ a_0 + \sum_{n=2}^{\infty} \frac{1}{1-n} (a_n \cos n\theta + b_n \sin n\theta) \right\},$$

(24)

where $v_c, v_s, a_0, a_n, b_n, n = 2, 3, \ldots, \infty$ can be determined from (20), (19b), (18), and (12b).
Consideration of Heat Source

After getting the real form thermal fundamental solution $v^*$ in (24), to obtain $w^*_y$ we let

$$[w^*_1] = r^2 \ln r \mathbf{a}_{11}(\theta) + r^2 \mathbf{a}_{21}(\theta), \quad [w^*_2] = r^2 \ln r \mathbf{a}_{12}(\theta) + r^2 \mathbf{a}_{22}(\theta),$$

where $\mathbf{a}_{11}(\theta), \mathbf{a}_{21}(\theta), \mathbf{a}_{12}(\theta), \mathbf{a}_{22}(\theta)$ are four vectors to be determined by the satisfaction of (5b). Substituting (25) and (24) into (5b), we get

$$2\mathbf{a}_{11}(\theta) + \mathbf{a}'_{12}(\theta) = v^*_x,$$

$$2\mathbf{a}_{12}(\theta) - \mathbf{a}'_{11}(\theta) = v^*_y,$$

$$[\mathbf{a}_{11}(\theta) + 2\mathbf{a}_{21}(\theta) + \mathbf{a}'_{22}(\theta)]\cos \theta + [\mathbf{a}_{12}(\theta) + 2\mathbf{a}_{22}(\theta) - \mathbf{a}'_{21}(\theta)]\sin \theta$$

$$= \mathbf{a}_0 + \sum_{n=2}^{\infty} \frac{1}{1-n^2} (\mathbf{a}_n \cos n\theta + \mathbf{b}_n \sin n\theta)$$

Apparently, equations (26) and (26)_2 can be satisfied if

$$\mathbf{a}_{11}(\theta) = \frac{1}{2} v^*_x, \quad \mathbf{a}_{12}(\theta) = \frac{1}{2} v^*_y.$$ (27)

With (27) and let

$$\mathbf{a}_{21}(\theta) = -\frac{1}{4} v^*_x + \frac{1}{3} \mathbf{a}_0 \cos \theta + \frac{1}{2} \mathbf{e}_1(\theta), \quad \mathbf{a}_{22}(\theta) = -\frac{1}{4} v^*_y + \frac{1}{3} \mathbf{a}_0 \sin \theta + \frac{1}{2} \mathbf{e}_2(\theta),$$ (28)

equation (26)_3 leads to

$$\mathbf{e}_1(\theta) + \frac{1}{2} \mathbf{e}'_2(\theta) = \sum_{n=2}^{\infty} \frac{1}{1-n^2} \{\cos(n+1)\theta \mathbf{a}_n + \sin(n+1)\theta \mathbf{b}_n\},$$

$$\mathbf{e}_2(\theta) - \frac{1}{2} \mathbf{e}'_1(\theta) = \sum_{n=2}^{\infty} \frac{1}{1-n^2} \{\sin(n+1)\theta \mathbf{a}_n - \cos(n+1)\theta \mathbf{b}_n\}. $$ (29)

From (29), we obtain

$$\mathbf{e}_1(\theta) = \sum_{n=2}^{\infty} \frac{2}{(n+3)(1-n^2)} \{\cos(n+1)\theta \mathbf{a}_n + \sin(n+1)\theta \mathbf{b}_n\},$$

$$\mathbf{e}_2(\theta) = \sum_{n=2}^{\infty} \frac{2}{(n+3)(1-n^2)} \{\sin(n+1)\theta \mathbf{a}_n - \cos(n+1)\theta \mathbf{b}_n\}. $$ (30)

Combining the results of (25), (27), (28) and (30), we have

$$[w^*_1] = \frac{1}{2} r^2 \ln r v^*_x + r^2 \left\{-\frac{1}{4} v^*_x + \frac{1}{3} \mathbf{a}_0 \cos \theta + \mathbf{w}_1\right\},$$

$$[w^*_2] = \frac{1}{2} r^2 \ln r v^*_x + r^2 \left\{-\frac{1}{4} v^*_y + \frac{1}{3} \mathbf{a}_0 \sin \theta + \mathbf{w}_2\right\},$$

where

$$\mathbf{w}_1 = \sum_{n=2}^{\infty} \frac{1}{(n+3)(1-n^2)} \{\cos(n+1)\theta \mathbf{a}_n + \sin(n+1)\theta \mathbf{b}_n\},$$

$$\mathbf{w}_2 = \sum_{n=2}^{\infty} \frac{1}{(n+3)(1-n^2)} \{\sin(n+1)\theta \mathbf{a}_n - \cos(n+1)\theta \mathbf{b}_n\}. $$ (31b)

3. Reduction of body integral to boundary integral

To avoid body discretization an exact transformation introduced in (4) is employed for the boundary element formulation. Using (4b) to find $v^*_i$ and $v^*_{i,k}$, and expressing the right hand side of (4a) in matrix form, we can prove that
\[
\int_{\Omega} \beta_{ij} u_{ij}^{*} (\xi, x) T(x) d\Omega = -\int_{\Gamma} [\nabla^{*} (\xi, x) \nabla^{\prime}(x) - \nabla^{**}(\xi, x) \tilde{k}(x) T(x)] d\Gamma,
\]

(32a)

where

\[
v^{*}(\xi, x) = \frac{1}{\pi} \text{Im} \left\{ A < k_{a}^{-1} (z_{a} - \tilde{z}_{a}) [\ln(z_{a} - \tilde{z}_{a}) - 1] > \beta_{d} \right\},
\]

\[
V^{**}(\xi, x) = \left[ v_{1}^{*}(\xi, x), \ v_{2}^{*}(\xi, x) \right],
\]

\[
\tilde{T}(x) = (k_{11} n_{1} + k_{21} n_{2}) T_{1} + (k_{12} n_{1} + k_{22} n_{2}) T_{2},
\]

\[
\tilde{k}(x) = \begin{bmatrix} k_{11} n_{1} + k_{21} n_{2} \\ k_{12} n_{1} + k_{22} n_{2} \end{bmatrix},
\]

(32b)

and

\[
v_{1}^{*}(\xi, x) = \frac{1}{\pi} \text{Im} \left\{ A < k_{a}^{-1} \ln(z_{a} - \tilde{z}_{a}) > \beta_{d} \right\},
\]

\[
v_{2}^{*}(\xi, x) = \frac{1}{\pi} \text{Im} \left\{ A < \mu_{a} k_{a}^{-1} \ln(z_{a} - \tilde{z}_{a}) > \beta_{d} \right\}.
\]

(32c)

If a multi-connected domain is considered, depending upon the location of the source point \( \xi \), the domain boundary can be assumed to be \( \Gamma = \Gamma_{0} + \Gamma_{1} + \ldots + \Gamma_{n} + \Gamma^{+} + \Gamma^{-} \), where \( \Gamma^{+} = \Gamma_{1}^{+} + \ldots + \Gamma_{n}^{+} \), \( \Gamma^{-} = \Gamma_{1}^{-} + \ldots + \Gamma_{n}^{-} \), and some portions of \( \Gamma_{i}^{\pm} \) may be split into \( \Gamma_{i}^{c} \) and \( \Gamma_{i}^{u} \), where the subscripts \( c \) and \( u \) denote, respectively, the cut and uncut portions (see Figures 1 (a)-(e)). Since \( t = -t^{*} \) and \( u = u^{*} \), it can be proved that

\[
\int_{\Gamma^{+} \setminus \Gamma^{-}} T^{*}(\xi, x) u(x) d\Gamma = \int_{\Gamma^{+} \setminus \Gamma^{-}} U^{*}(\xi, x) t(x) d\Gamma = 0, \quad \text{for all } \xi,
\]

(33)

where \( \xi_{0} \) denotes the point lying on the outer boundary of the body. Note that the first equation of (33) is always valid no matter which branch is selected for the calculation of logarithmic function. The second equation is valid only when the branch is chosen such that it will not cut the body. With this concern, the branch cut is usually suggested to be directed in the outward normal direction, i.e., a branch leading to

\[
\theta_{j} + \pi / 2 \leq \theta_{j} \leq \theta_{j} + 5\pi / 2,
\]

(34)

where \( \theta_{j} \) is the angle of tangential direction of the boundary node \( j \). However, when the source point \( \xi \) locates inside the body, no such branch cut exists.

If the branch cut for the principal value of the logarithmic function is selected such that \( \ln z = \ln r + i\theta \) and \(-\pi \leq \theta \leq \pi\), it can be shown that

\[
\int_{\Gamma^{+} \setminus \Gamma^{-}} [v^{*}(\xi, x) \nabla^{\prime}(x) - \nabla^{**}(\xi, x) \tilde{k}(x) T(x)] d\Gamma
\]

\[
= a_{1} v_{1} - a_{2} [v_{1}, v_{2}] k_{0}, \quad \text{when } \xi \neq \xi_{0},
\]

(35a)

where

\[
v_{1} = -2 \text{Re} \left\{ A < k_{a}^{-1} > \beta_{d} \right\}, \quad v_{2} = -2 \text{Re} \left\{ A < k_{a}^{-1} \mu_{a} > \beta_{d} \right\}, \quad k_{0} = \begin{bmatrix} k_{21} \\ k_{22} \end{bmatrix},
\]

(35b)

\[
a_{1} = \sum_{i=1}^{n_{1}} \int_{L_{i}} (k_{21} T_{1}(x) + k_{22} T_{2}(x)) dx, \quad a_{2} = \sum_{i=1}^{n_{1}} \int_{L_{i}} T(x) dx.
\]

In (35b), the integration variable \( x \) is a local variable with the origin at the source point \( \xi \), and \( L_{i}, i = 1,\ldots,n \) are the horizontal lines on the left hand side of the source point \( \xi \). For example, if
$\xi = \xi_A$, then $n_i = 1$ and $L_i = \Gamma_i^-$ (Figure 1 (a)); if $\xi = \xi_B$, then $n_i = 2$ and $L_i = \Gamma_i^-$, $L_2 = \Gamma_2^-$ (Figure 1 (b)); if $\xi = \xi_C$, then $n_i = 2$ and $L_i = \Gamma_i^-$, $L_2 = \Gamma_2^-$ (Figure 1(c)); if $\xi = \xi_D$, then $n_i = 1$ and $L_i = \Gamma_i^-$ (Figure 1 (d)); if $\xi = \xi_E$, as shown in (33) the line integral along $\Gamma^+ + \Gamma^-$ will be zero.

![Figure 1 A multi-connected domain.](image)

From the above discussion we know that due to the discontinuity across the branch cut selected for the logarithmic function $\ln(z_a - \hat{z}_a)$, to evaluate the boundary integral (32a) an extra line integral (35a) acrossing the domain surface is required for the problem with multi-connected domain. Therefore, even only the boundary integral is involved in (32a) we still need to provide the temperature information inside the domain, which will reduce the value of BEM. To avoid the problem induced by the complex logarithmic function, by solving (4b) and using the Stroh's identities to convert complex form to real form, a continuous real form solution for $v_i^*$ has been obtained in (24), and hence we can get a truly boundary integral by the conversion given in (4) and (5). The derivatives of $v_i^*$ and $w_i^*$, which are required in the boundary element formulation, can then be derived from (24) and (31) as
\[ v^*_1 = [(\ln r + 1)v_1(\theta) + v_2(\theta)] \cos \theta - [(\ln r)_{\theta}^\prime(\theta) + v_2^\prime(\theta)] \sin \theta, \]
\[ v^*_2 = [(\ln r + 1)v_1(\theta) + v_2(\theta)] \sin \theta + [(\ln r)_{\theta}^\prime(\theta) + v_2^\prime(\theta)] \cos \theta, \]
\[ v^*_{11} = \frac{1}{r} \left\{ v_1 - v_1^* \sin 2\theta + (v_2 + v_2^*) \sin^2 \theta \right\}, \]
\[ v^*_{12} = \frac{1}{r} \left\{ v_1 \cos 2\theta - (v_2 + v_2^*) \cos \theta \sin \theta \right\}, \]
\[ v^*_{22} = \frac{1}{r} \left\{ v_1 + v_1^* \sin 2\theta + (v_2 + v_2^*) \cos^2 \theta \right\}, \]
\[ w^*_{i1,1} = r \left\{ \cos \theta \ln r v_1 + \frac{1}{6} (3 + \cos 2\theta) a_0 + 2 \cos \theta w_1 - \sin \theta w_1^\prime \right\}, \]
\[ w^*_{i1,2} = r \left\{ \sin \theta \ln r v_1 + \frac{1}{6} \sin 2\theta a_0 + 2 \sin \theta w_1 + \cos \theta w_1^\prime \right\}, \]
\[ w^*_{i2,1} = r \left\{ \cos \theta \ln r v_1 + \frac{1}{6} \sin 2\theta a_0 + 2 \cos \theta w_2 - \sin \theta w_2^\prime \right\}, \]
\[ w^*_{i2,2} = r \left\{ \sin \theta \ln r v_1 + \frac{1}{6} (3 - \cos 2\theta) a_0 + 2 \sin \theta w_2 + \cos \theta w_2^\prime \right\}. \quad (36a) \]

4. Application to the problems with multi-connected domain

To demonstrate the application to the problems with multi-connected domain, several numerical examples are implemented by using the boundary element method developed in this project, in which the thermal fundamental solutions are the real form solution presented in the previous sections. In the analyses of all numerical examples, the Fourier series with total 12 coefficients was employed (i.e. \( n \) from 2 to 13), where the integral of the Fourier analysis was computed by the 128-point Gaussian quadrature rule. As the pre-process of the subsequent thermoelastic analysis of each example, the associated thermal field was firstly solved to provide temperature as well as its spatial gradients at all boundary nodes. In this study, the thermal data for example 1 were obtained from exact solution (Hwu, 2010), and that for example 2 were calculated by finite element software ANSYS. With the obtained thermal data on boundary, the boundary integral equation (2) with (4) or (5) can thus be solved to compute all boundary unknowns.

Example 1. An infinite plate with an elliptic hole subjected to uniform heat flow

To verify our results by comparing with the analytical solutions obtained for the infinite domain (Hwu, 2010), we firstly consider an infinite orthotropic elastic plate with an insulated elliptical hole subjected to a uniform heat flow \( h_0 = 100 \text{ W/m}^2 \) in the direction of positive \( x_2 \)-axis (see Figure 2(a)). The half lengths of the major and minor axes of the elliptical hole are \( a = 0.5 \text{ m} \) and \( b = 0.1 \text{ m} \). Plane stress condition is assumed for the present example. The elastic and thermal properties of the plate are

\[
\begin{align*}
C_{11} &= 58.74 \text{ GPa}, \quad C_{22} = C_{33} = 23.35 \text{ GPa}, \quad C_{23} = 6.55 \text{ GPa}, \\
C_{12} &= C_{13} = 7.48 \text{ GPa}, \quad C_{44} = C_{55} = C_{66} = 9.7 \text{ GPa}, \\
\alpha_{11} &= 6.3 / {^\circ}\text{C}, \quad \alpha_{22} = \alpha_{33} = 20 / {^\circ}\text{C}, \\
k_{11} &= 3.46 \text{ W/m}^\circ{\text{C}}, \quad k_{22} = k_{33} = 0.35 \text{ W/m}^\circ{\text{C}}.
\end{align*}
\]

To simulate the infinite plate, in BEM modeling we consider a square plate with side length \( \ell = 20 \text{ m} \) and employ 160 linear elements with total 164 nodes as shown in Figure 2(b). The
results of hoop stress $\sigma_{ss}$ along the hole boundary calculated by the analytical solutions as well as BEM with the proposed real form fundamental solution and complex form with or without extra line integral are all presented and compared. From Table 1 we see that both of the BEM results obtained from complex form with extra line integral and real form without extra line integral are in good agreement with the analytical solutions. On the other hand, the results from complex form without extra line integral are totally wrong. In this Table and all the following figures, variable $\psi$ is a parameter used to denote the point on the elliptical or circular hole boundary, which satisfies $x_i = a \cos \psi, \quad x_2 = b \sin \psi$.

Figure 2. (a) An infinite plate with an elliptical hole subjected to uniform heat flow, (b) plate geometry and BEM mesh.

Table 1. Variation of hoop stress $\sigma_{ss}$ with respect to angular parameter $\psi$ along the elliptical hole boundary.

<table>
<thead>
<tr>
<th>$\psi$</th>
<th>$\sigma_{ss}$ (MPa)</th>
<th>$\sigma_{ss}$ (MPa)</th>
<th>$\sigma_{ss}$ (MPa)</th>
<th>$\sigma_{ss}$ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-5.49E-15</td>
<td>-1.12E-07</td>
<td>-9.99E-08</td>
<td>-986</td>
</tr>
<tr>
<td>11</td>
<td>8.54</td>
<td>8.50</td>
<td>8.42</td>
<td>1507</td>
</tr>
<tr>
<td>22</td>
<td>11.75</td>
<td>11.12</td>
<td>11.60</td>
<td>2036</td>
</tr>
<tr>
<td>32</td>
<td>13.58</td>
<td>13.09</td>
<td>13.41</td>
<td>1996</td>
</tr>
<tr>
<td>41</td>
<td>14.88</td>
<td>14.64</td>
<td>14.86</td>
<td>1787</td>
</tr>
<tr>
<td>53</td>
<td>16.32</td>
<td>16.27</td>
<td>16.34</td>
<td>1454</td>
</tr>
<tr>
<td>61</td>
<td>17.10</td>
<td>17.22</td>
<td>17.22</td>
<td>1198</td>
</tr>
<tr>
<td>71</td>
<td>17.84</td>
<td>18.01</td>
<td>17.92</td>
<td>923</td>
</tr>
<tr>
<td>81</td>
<td>18.28</td>
<td>18.52</td>
<td>18.41</td>
<td>619</td>
</tr>
<tr>
<td>90</td>
<td>18.41</td>
<td>18.69</td>
<td>18.53</td>
<td>329</td>
</tr>
</tbody>
</table>
To check the applicability of both analytical solution and BEM to the MEE materials, the orthotropic elastic material considered in this example is then replaced by the MEE material made by mixing piezoelectric BaTiO$_3$ and piezomagnetic CoFe$_2$O$_4$, whose properties are

\[
C_{ij}^{E,H} = \begin{bmatrix}
  C_{11}^{E,H} & C_{21}^{E,H} & C_{31}^{E,H} \\
  C_{21}^{E,H} & C_{22}^{E,H} & C_{23}^{E,H} \\
  C_{31}^{E,H} & C_{23}^{E,H} & C_{33}^{E,H}
\end{bmatrix} = \begin{bmatrix}
  126 GPa & 226 GPa \\
  44.2 GPa & 216 GPa \\
  124 GPa & 50.5 GPa
\end{bmatrix},
\]

\[
e_{ij}^{E,H} = \begin{bmatrix}
  e_{11}^{E,H} & e_{21}^{E,H} & e_{31}^{E,H} \\
  e_{21}^{E,H} & e_{22}^{E,H} & e_{23}^{E,H} \\
  e_{31}^{E,H} & e_{23}^{E,H} & e_{33}^{E,H}
\end{bmatrix} = \begin{bmatrix}
  2.2 C/m^2 & 9.3 C/m^2 \\
  5.8 C/m^2 & 11.2 \times 10^{-9} C^2/Nm^2 \\
  12.6 \times 10^{-9} C^2/Nm^2 & 11.2 \times 10^{-9} C^2/Nm^2
\end{bmatrix},
\]

\[
\omega_{ij}^{E,H} = \begin{bmatrix}
  \omega_{11}^{E,H} & \omega_{21}^{E,H} & \omega_{31}^{E,H} \\
  \omega_{21}^{E,H} & \omega_{22}^{E,H} & \omega_{23}^{E,H} \\
  \omega_{31}^{E,H} & \omega_{23}^{E,H} & \omega_{33}^{E,H}
\end{bmatrix} = \begin{bmatrix}
  290.1 N/Am & 349.9 N/Am \\
  275 N/Am & 92.7 \times 10^{-9} C^2/Nm^2 \\
  92.7 \times 10^{-9} C^2/Nm^2 & 92.7 \times 10^{-9} C^2/Nm^2
\end{bmatrix},
\]

\[
m_{ij}^{E,H} = \begin{bmatrix}
  m_{11}^{E,H} & m_{21}^{E,H} & m_{31}^{E,H} \\
  m_{21}^{E,H} & m_{22}^{E,H} & m_{23}^{E,H} \\
  m_{31}^{E,H} & m_{23}^{E,H} & m_{33}^{E,H}
\end{bmatrix} = \begin{bmatrix}
  5.367 \times 10^{-12} Ns/VC & 2737.5 \times 10^{-12} Ns/VC \\
  2.97 \times 10^{-12} Ns/VC & 2.97 \times 10^{-12} Ns/VC \\
  2.97 \times 10^{-12} Ns/VC & 2.97 \times 10^{-12} Ns/VC
\end{bmatrix},
\]

\[
e'e_{ij}^{E,H} = \begin{bmatrix}
  e'_{11}^{E,H} & e'_{21}^{E,H} & e'_{31}^{E,H} \\
  e'_{21}^{E,H} & e'_{22}^{E,H} & e'_{23}^{E,H} \\
  e'_{31}^{E,H} & e'_{23}^{E,H} & e'_{33}^{E,H}
\end{bmatrix} = \begin{bmatrix}
  50 W/m C & 75 W/m C \\
  0.835 \times 10^{-4} Ns/C & 0.835 \times 10^{-4} Ns/C \\
  0.835 \times 10^{-4} Ns/C & 0.835 \times 10^{-4} Ns/C
\end{bmatrix},
\]

\[
k_{11} = k_{22} = 50 W/m°C, k_{33} = 75 W/m°C,
\]

\[
\alpha_{11} = \alpha_{33} = 1.025°C/C, \quad \alpha_{22} = 0.92°C/C,
\]

\[
\beta_{12} = 25 \times 10^{-6} C^2/Cm^2, \quad \beta_{23} = 5.19 \times 10^{-6} N/Am°C.
\]

Plane stress with $E_3=0$ and $H_3=0$ is assumed for the following presented results. To get more accurate solutions, the mesh along the hole boundary was made twice as the one for the orthotropic elastic material. Figures 3(a), 3(b) and 3(c) show, respectively, the results of hoop stress $\sigma_{ss}$, electric displacement $D_1$, and magnetic induction $B_2$ along the hole boundary, in which the real form fundamental solution is used in BEM. As can be observed from this Figure, the BEM results are indeed in good agreements with the analytical solutions. Note that in BEM modelling of this example, in addition to the exclusion of the rigid body motion, the electric and magnetic potentials at the left bottom corner are prescribed to be zero.
Example 2. A hollow disk under prescribed temperature and heat generation rate

After the verification by analytical solution for the problem with infinite domain, we now verify our solutions by the problem with finite domain through the commercial finite element software ANSYS. As shown in Figure 4, a hollow disk with outer radius $r_o=1$ m and inner radius $r_i=0.5$ m is considered, where the outer and inner surfaces are prescribed with 0°C and 100°C, respectively. The outer surface is fully constrained ($u_1=u_2=u_3=u_4=0$), whereas the inner side is free of traction ($t_1=t_2=t_3=0$) and electric potential ($u_4=0$). To study the influence of the heat generation rate, both $\gamma=0$ and $\gamma=1000$ W/m$^3$ inside the hole are analyzed. The plane stress and open circuit condition, i.e., $\sigma_3=D_3=0$, is assumed.

Since ANSYS does not provide the function for the general MEE materials, for the purpose of comparison only special cases of MEE such as piezoelectric materials are considered. The
properties of piezoelectric materials are

\[ C_{11}^{E,H} = C_{33}^{E,H} = 166 \text{ GPa}, \quad C_{12}^{E,H} = C_{23}^{E,H} = 162 \text{ GPa}, \quad C_{13}^{E,H} = C_{23}^{E,H} = 78 \text{ GPa}, \quad C_{13}^{E,H} = 77 \text{ GPa}, \]

\[ C_{44}^{E,H} = C_{55}^{E,H} = 43 \text{ GPa}, \quad C_{11}^{E,H} = C_{22}^{E,H} = 44.5 \text{ GPa}, \quad C_{j}^{E,H} = C_{j}^{E,H}, \quad i, j = 1,2,\ldots,6, \]

\[ e_{21}^{H} = e_{23}^{H} = -4.4 \text{ C/m}^2, \quad e_{32}^{H} = 18.6 \text{ C/m}^2, \quad e_{34}^{H} = e_{16}^{H} = 11.6 \text{ C/m}^2, \]

\[ \alpha_{11}^{E,H} = \alpha_{33}^{E,H} = 11.2 \times 10^{-9} \text{C/Nm}^2, \quad \alpha_{55}^{E,H} = 12.6 \times 10^{-9} \text{C/Nm}^2, \]

\[ k_{11} = k_{22} = k_{33} = 2.5 \text{W/m}^\circ\text{C}. \]

\[ \alpha_{11} = 15.7/\circ\text{C}, \quad \alpha_{22} = 6.4/\circ\text{C}, \quad \beta_{12} = 2 \times 10^{-4} \text{C}/\text{Cm}^2. \]

To generate generally anisotropic properties, in the present example the principle material direction is rotated 30° counterclockwise about the \(x_3\)-axis. After the standard convergence test, 16200 elements with 49320 nodes were employed in ANSYS, and 256 linear elements with 256 nodes were employed in BEM modelling. Figure 5 shows the results of displacement \(u_2\), electric potential \(u_4\), stress \(\sigma_{22}\) and electric displacement \(D_2\) along the inner circle \(r=0.75\text{ m}\). From this Figure we see that the BEM solutions agree fairly well with the ANSYS solutions, and the non-zero heat generation rate will in general increase the magnitude of displacement, stress and electric displacement. Only the values of electric potential \(u_4\) remain almost unchanged for different heat generation rate.

Figure 5. (a) Displacement \(u_2\), (b) electric potential \(u_4\), (c) stress \(\sigma_{22}\), and (d) electric displacement \(D_2\) along the inner circle \(r=0.75\text{ m}\).
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References
Nowacki, W., 1962, *Thermoelasticity*, Addison-Wesley, Reading, Mass..