黏彈性力學之邊界元素設計及其應用(1/3)
Boundary Element Design for Viscoelasticity and its Applications (1/3)

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The main purpose of this project is the establishment of boundary element for the problems of interfaces/holes/inclusions/cracks in linear anisotropic viscoelastic materials. To achieve this goal, in the first year we re-developed the extended Stroh formalism for linear anisotropic viscoelasticity and studied the usable numerical inversion of Laplace transform. Through the re-derivation of extended Stroh formalism, a correspondence principle has been obtained not only for the traditional displacements and stresses but also for the material eigenvector matrices and complex functions. Also, three different possible methods for the numerical Laplace inversion have been studied and a simple example is presented in this report.

Keywords: viscoelasticity, boundary element method, anisotropic elasticity, interface, hole, inclusion, crack, fracture analysis

According to our original plan, in the first we re-developed the extended Stroh formalism for linear anisotropic viscoelasticity and studied the usable numerical inversion of Laplace transform. The preliminary results of these two topics are as follows.

1. Extended Stroh formalism for linear anisotropic viscoelasticity

In a fixed Cartesian coordinate system, \( x_i, \ i=1,2,3 \), the constitutive law for the homogeneous linear anisotropic viscoelastic materials, the strain-displacement relation for the small deformation, and the equilibrium equation for the static loading condition can be written as

\[
\sigma_{ij} = C_{ijkl} \ast \mathbf{e}_{kl}, \quad \mathbf{e}_{ij} = (u_{ij} + u_{ji})/2, \quad \sigma_{ij,j} = 0, \quad (1)
\]

where \( i,j,k,l = 1,2,3 \); the repeated indices imply summation and a comma stands for differentiation; \( u_i \equiv u_i(x,t), \quad \mathbf{e}_{ij} \equiv \mathbf{e}_{ij}(x,t) \), and \( \sigma_{ij} \equiv \sigma_{ij}(x,t) \) are, respectively, the displacements, strains, and stresses in which \( x = (x_1,x_2,x_3) \) denotes spatial coordinate and \( t \) denotes time; \( C_{ijkl} = C_{ijkl}(t) \) is the elastic stiffness tensor whose components are also known to be the relaxation functions of the viscoelastic materials, and the symmetry of stress and strain imply \( C_{ijkl}(t) = C_{jikl}(t) = C_{iklj}(t) \); the operator \( \ast \) denotes the Stieltjes convolution, i.e.,

\[
\varphi(t) \ast \varphi'(t) = \int_{-\infty}^{t} \varphi(t - \tau) \varphi'(\tau) \, d\tau = \varphi(t)\varphi(0) + \int_{0}^{t} \varphi(t - \tau) \frac{\partial \varphi(\tau)}{\partial \tau} \, d\tau, \quad (2)
\]

where the second equality is obtained under the condition that \( \varphi(t) = 0 \) when \( t < 0 \).

Combining the three sets of equations given in (1) and using the symmetric property of relaxation functions, we get

\[
C_{ijkl} \ast d\mathbf{u}_{k,ij} = 0. \quad (3)
\]

For two-dimensional deformation in which \( u_k, k = 1,2,3 \), depend on \( x_1 \) and \( x_2 \) only, a general
solution for \( u_k \) can be assumed as

\[
    u_k(x,t) = a_k(t) \ast df(z,t),
\]

or

\[
    u_k(x,t) = \int_{-\infty}^{t} a_k(t - \tau) df(z,\tau) = a_k(t) f(0) + \int_{0}^{t} a_k(t - \tau) \frac{\partial f(z,\tau)}{\partial \tau} d\tau,
\]

where

\[
    z = x_1 + \mu x_2.
\]

Substituting (4) into (3), we get

\[
    C_{ijk} \ast da_k \ast df'' + (C_{ijk} + C_{ijk}) \ast da_k \ast d[\mu f'] + C_{ik} \ast da_k \ast d[\mu^2 f''] = 0,
\]

in which

\[
    f'' = \frac{\partial^2 f(z,t)}{\partial z^2}.
\]

If \( \mu \) is independent of time, which has been proved in this study for the standard linear viscoelastic materials, (5) will lead to

\[
    D_k \ast da_k \ast df'' = 0, \quad \text{or,} \quad \int_{-\infty}^{t} \int_{-\infty}^{t-\tau} D_k(t - \tau - \omega) da_k(\omega) df'(z,\tau) = 0,
\]

where

\[
    D_k = C_{ik} + \mu(C_{ik} + C_{ik}) + \mu^2 C_{ik}.
\]

By taking the Laplace transform of (6a), we get

\[
    s^2 \tilde{D}_k a_k \tilde{f}'' = 0,
\]

where the over breve \( \tilde{\cdot} \) denotes the Laplace transform defined by

\[
    \tilde{f}(s) = \int_{0}^{\infty} f(t)e^{-st}dt \equiv L\{f(t)\}
\]

In order to have equation (7) be satisfied for any \( s \) and \( \tilde{f}'' \), the value of \( \tilde{D}_k a_k \) should be zero. Written in matrix form, we have

\[
    {Q + \mu(R + R^T) + \mu^2T} \tilde{a} = 0,
\]

where

\[
    Q_{ik} = sC_{1ik}(s), \quad R_{ik} = sC_{1ik}(s), \quad T_{ik} = sC_{1ik}(s), \quad i,k = 1,2,3.
\]

Note that at this stage the results of \( \mu \) and \( \tilde{a} \) will not be influenced by the multiplication of the transform variable \( s \) for \( Q_{ik}, R_{ik} \) and \( T_{ik} \) defined in (9b). Here, the appearance of the multiplication factor \( s \) is totally due to the correspondence principle stated later in this Section.

After getting \( \mu \) and \( \tilde{a} \) from (9), \( a_k \) (whose \( k \)th component is \( a_k \)) can be obtained by the Laplace inversion of \( \tilde{a} \). Since (9) is exactly the same as that of anisotropic elastic material, which have been proved to have three pairs of complex conjugate \( \mu \) to guarantee the positive strain energy density (Ting, 1996; Hwu, 2010), we may let

\[
    \mu_{k+3} = \overline{\mu}_k, \quad a_{k+3} = \overline{a}_k, \quad k = 1,2,3,
\]

where the over-bar denotes the complex conjugate. If the stress function \( \phi_i,i = 1,2,3 \), are introduced as

\[
    \sigma_{i1} = -\phi_{i2}, \quad \sigma_{i2} = \phi_{i1},
\]

substitution of (4a) into (1) will lead to

\[
    \phi_i = b_i \ast df'(z), \quad \text{or} \quad \phi_i(x,t) = \int_{-\infty}^{t} b_i(t - \tau) df'(z,\tau),
\]

where
\[ b_i = (C_{12k1} + \mu C_{12k2}) a_k = \frac{-1}{\mu} (C_{11k1} + \mu C_{11k2}) a_k. \] 

By taking the Laplace transform of (12b), the results can be written in matrix form as

\[ \mathbf{b} = (\mathbf{R} + \mu \mathbf{T}) \mathbf{a} = \frac{1}{\mu} (\mathbf{Q} + \mu \mathbf{R}) \mathbf{a}. \] 

Combining the results of (4a) and (12a), the general solutions in time domain for the two-dimensional linear anisotropic viscoelasticity can be written in the matrix forms as

\[ \mathbf{u}(x,t) = 2 \text{Re} \{ \mathbf{A}(t)^* \mathbf{d}(z,t) \}, \quad \mathbf{\phi}(x,t) = 2 \text{Re} \{ \mathbf{B}(t)^* \mathbf{d}(z,t) \}, \]

and

\[ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \mathbf{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}, \quad \mathbf{f}(z,t) = \begin{bmatrix} f_1(z_1,t) \\ f_2(z_2,t) \\ f_3(z_3,t) \end{bmatrix}, \]

\[ \mathbf{A}(t) = [\mathbf{a}_1(t) \quad \mathbf{a}_2(t) \quad \mathbf{a}_3(t)], \quad \mathbf{B}(t) = [\mathbf{b}_1(t) \quad \mathbf{b}_2(t) \quad \mathbf{b}_3(t)], \]

where\[ z_k = x_1 + \mu_k x_2, \quad k = 1,2,3, \]

Correspondence principle

Taking the Laplace transform of (1) gives

\[ \bar{\sigma}_{ij}(s) = s \bar{C}_{ijkl}(s) \bar{\epsilon}_{kl}(s), \quad \tilde{\epsilon}_{ij}(s) = \frac{1}{2} \{ \bar{u}_{ij,s}(s) + \bar{u}_{ij,s}(s) \}, \quad \bar{\sigma}_{ij,s}(s) = 0, \]

which are identical to the basic equations of linear anisotropic elasticity. Thus, the viscoelastic solutions in Laplace domain can be obtained directly from the solutions of the corresponding elastic problems with the replacement of the elastic stiffness tensor \( C_{ijkl} \) by \( s \bar{C}_{ijkl}(s) \), if the boundary of a viscoelastic body is invariant with time. This statement is the so-called correspondence principle between linear elasticity and linear viscoelasticity (Read, 1950; Sips, 1951; Brull, 1953; Lee, 1955) and is applicable to anisotropic viscoelastic materials.

By using the correspondence principle and Stroh formalism for two-dimensional linear anisotropic elasticity (Ting, 1996; Hwu, 2010), the general solutions satisfying the 15 partial differential equations (15) can be written as

\[ \bar{\mathbf{u}}(x,s) = 2 \text{Re} \{ \mathbf{A}_s(s) \mathbf{f}_s(z,s) \}, \quad \bar{\mathbf{\phi}}(x,s) = 2 \text{Re} \{ \mathbf{B}_s(s) \mathbf{f}_s(z,s) \}, \]

and

\[ \bar{\mathbf{u}} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{bmatrix}, \quad \bar{\mathbf{\phi}} = \begin{bmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \\ \bar{\phi}_3 \end{bmatrix}, \quad \mathbf{f}_s(z,s) = \begin{bmatrix} f_{1,s}(z_1,s) \\ f_{2,s}(z_2,s) \\ f_{3,s}(z_3,s) \end{bmatrix}, \]

\[ \mathbf{A}_s(s) = [\mathbf{a}_1^s(s) \quad \mathbf{a}_2^s(s) \quad \mathbf{a}_3^s(s)], \quad \mathbf{B}_s(s) = [\mathbf{b}_1^s(s) \quad \mathbf{b}_2^s(s) \quad \mathbf{b}_3^s(s)], \]

\[ z_k = x_1 + \mu_k x_2, \quad k = 1,2,3, \]

\( \bar{\mathbf{u}} \) and \( \bar{\mathbf{\phi}} \) are the displacement and stress function vectors in Laplace domain, and \( \bar{\phi}_i \) is related to the stresses in Laplace domain by

\[ \bar{\sigma}_{11} = -\bar{\phi}_{12}, \quad \bar{\sigma}_{12} = \bar{\phi}_{11}. \]

\( \mathbf{f}_s(z) \) is a function vector composed of three holomorphic complex functions \( f^s_{\alpha}(z_\alpha) \), \( \alpha = 1,2,3 \), which will be determined through the satisfaction of the boundary conditions. \( \mu_{\alpha} \) and \( \mathbf{a}_{\alpha}, \mathbf{b}_{\alpha} \) are the material eigenvalues and eigenvectors which can be determined by the following
eigenrelations:
\[ N\xi = \mu\xi, \quad (18a) \]
where \( N \) is a 6×6 fundamental elasticity matrix and \( \xi \) is a 6×1 column vector defined by
\[ N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}, \quad \xi = \begin{bmatrix} a \\ b \end{bmatrix}, \quad (18b) \]
and
\[ N_1 = -T^{-1}R^T, \quad N_2 = T^{-1} = N_2^T, \quad N_3 = RT^{-1}R^T - Q = N_3^T. \quad (18c) \]
In the above, the superscript \( T \) denotes the transpose of a matrix. \( Q, R, T \) are three 3×3 real matrices defined in (9b). The subscript or superscript \( s \) denotes the value in Laplace domain.

By taking the Laplace transform of (14a) and comparing the results with (16a), we see that
\[ \tilde{A} = A_s, \quad \tilde{B} = B_s, \quad s\tilde{f} = f_s, \quad (19) \]
and the eigenrelation (18) can be derived from (13).

2. Numerical inversion of Laplace transform

In the extended Stroh formalism for linear viscoelasticity, some values in real time domain are determined by numerical inversion of Laplace transform. To see how to perform the Laplace inversion numerically, three methods are stated briefly in this section.

Schapery’s collocation method

Schapery’s collocation method (Schapery, 1962) is the most common technique employed to deal with viscoelastic problems. If a function \( f(t) \) is represented by
\[ f(t) = A + Bt + \sum_{k=1}^{m} a_k e^{-b_k t}, \quad (20) \]
where \( A, B, a_k \) and \( b_k \) are constant in time and \( m \) is an arbitrary number of terms in the exponential series, the Laplace transform of (20) gives
\[ \tilde{f}(s) = s + \sum_{k=1}^{m} \frac{a_k s}{s + b_k} \quad (21) \]
If the function values in Laplace domain are calculated for a sequence of values \( s = s_1, s_2, ..., s_{m+2} \), and the powers of the exponential functions in (20), \( b_k \), are assumed to be equal to \( s_k \), then a system of \( m+2 \) linear algebraic equations can be set to solve the \( m+2 \) unknown constants \( A, B \) and \( a_k, k = 1,2, ..., m \). The function values in time domain can therefore be calculated through (20).

Inversion through curve fitting of particular functions

If a function in Laplace transform domain can be well fitted by the following functions
\[ f_i(s) = a_1 + a_2 \log_{10} s + a_3 (\log_{10} s)^2 + \frac{a_4}{(s + a_5)} + \frac{a_6}{(s + a_7)^2} + \frac{a_8}{(s + a_9)^3}, \quad (22) \]
where the coefficients \( a_i, i = 1,2,3,...,9 \), are determined through the least square method. Its associated value in time domain can be calculated by
where $C \approx 0.5772$ is the Euler constant (Kreyszig, 1999).

**Stehfest method**

Stehfest (1970) considered the expectation of $f(t)$ with respect to a certain probability density function, and it can be formulated as

$$f_s(t) = \frac{\ln 2}{t} \sum_{i=1}^{n} v_i \tilde{f}(\frac{\ln 2}{t})$$

with

$$v_i = (-1)^{n/2+i} \sum_{k=\text{floor}(\frac{i+1}{2})}^{\text{min}(n/2)} \frac{k^{n/2} (2k)!}{(n/2-k)! (k-1)! (i-k)! (2k-i)!}$$

where $f_s(t)$ is the approximation of $f(t)$, $n$ is a natural number and must be a power of 2, and the operator “floor” gives the greatest integer less than or equal to its argument, while “min” yields the smaller of its two arguments. The only task is to make a decision of the natural number $n$ when we perform this method. Stehfest said that theoretically $f_s(t)$ becomes the more accurate the greater $n$, however, if $n$ becomes too large rounding errors will worsen the results due to the greater and greater absolute values of $v_i$. Therefore, a convergent test should be done with increasing $n$. The Stehfest method had been built in web resources of Wolfram Research Inc.

**Convergence study of Laplace Inversion (Schapery’s collocation method)**

Since the proposed BEM will be performed in Laplace transform domain, the numerical inversion of Laplace transform stated above is necessary to get the physical responses in time domain. From (20) and (21) we see that the approximation of the numerical inversion depends upon the selection of the transformed variables. To know how to select the transformed variables properly, before executing the BEM a simple example on the Laplace inversion of shear relaxation function $G(t)$ is considered as follows. If $G(t)$ is known to be

$$G(t) = G_\infty + (G_0 - G_\infty) e^{-t/\tau}, \quad (25)$$

where $G_0$ and $G_\infty$ are, respectively, the shear moduli at the initial and final states, and parameter $\tau$ is the relaxation time that determines the rate of decay, its Laplace transform is

$$\tilde{G}(s) = \frac{G_\infty}{s} + \frac{(G_0 - G_\infty)}{s + 1/\tau}. \quad (26)$$

By taking (26) as a known function, its inversion (25) may be approximated by (20) through suitable selection of the transformed variables $s_1, s_2, \ldots, s_{m+2}$ to solve the unknown constants $A, B$ and $a_k, k = 1, 2, \ldots, m$ ($b_k$ are assumed to be equal to $s_k$). With this understanding, the transformed variables $s_1, s_2, \ldots, s_{m+2}$ are suggested to be selected as
\[ s_k = 10^{x_k}, \quad k = 1, 2, \ldots, m + 2, \quad (27a) \]

where

\[ x_1 = -2 \log_{10} \tau, \quad \Delta x = \frac{4 \log_{10} \tau}{m + 1}, \quad (27b) \]

\[ x_k = x_1 + (k - 1) \Delta x, \quad k = 2, 3, \ldots, m + 2. \]

This selection will ensure \( s_1 < 1/\tau < s_{m+2} \). Figure 1 shows the comparison for the case with \( G_0 / G_\infty = 2.19 \) and \( \tau = 10 \) sec. From this figure we see that the approximation will be improved when more transformed variables are used, i.e., for larger \( m \). When the exact solution can be described by the exponential function like (25) considered in this example, if the powers of the exponential function can be covered by the selected transformed variables like \( m=3 \) of Figure 1 whose \( s_2 = 0.1 = 1/\tau \) no matter how small of \( m \) the numerical inversion will always offer the best approximation.

![Figure 1. Comparison of the numerical Laplace inversion.](image)

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**References**


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   本年度建立適用於異向性黏彈性力學之史蹉公式，同時探討拉氏反運算數值方法。 經由史蹉公式之推演我們將傳統之黏彈對等原則擴充至材料特徵矩陣及其相關之複變函數。另 外提出三個拉氏反運算數值方法，並以一簡單例證說明拉氏反運算數值方法之數值收斂情形。 這些計算及驗證工作皆如原預期得到適當之結果，相信這些結果對未來黏彈性材料邊界元素之建立必有實質之幫助。