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Extended Failure Analysis of Interface Corners (1/3)

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Abstract
In our previous project [NSC 91-2212-E-006-137], we have derived the analytical closed-form solution for the order of stress singularity of interface corner. However, only understanding the stress singularity is not enough to determine the fracture behavior of interface corners. Consequently, a unified definition of stress intensity factors corresponding to the most critical singular order was proposed in our last project [NSC 95-2221-E-006-144-MY3]. This definition is simultaneously suitable for cracks, corners, interface cracks, and interface corners. In that project, we also found that an interface corner may possess more than one singular orders in which the most critical one may not correspond to all three modes of stress intensity factors. Therefore, if only the most critical singular order is concerned we cannot completely understand the fracture behavior of interface corners. To obtain full modes of stress intensity factors of interface corners, this mid-term report will discuss which kind of interface corner possesses the most critical singular order that doesn't correspond to full modes of stress intensity factors and present a complementary definition for stress intensity factors to this kind of interface corners. Besides, to understand the fracture mechanism of viscoelastic interface corners more clearly, by combining the Stroh formalism and the correspondence principle between linear elasticity and linear viscoelasticity in this project we establish a mathematical model to describe the general solutions of displacements and stresses for viscoelastic materials. Furthermore, by combining these general solutions and the achievement of the project [NSC 91-2212-E-006-137] this report also present the analytical solution for the orders of stress singularity of the interface corners composed of viscoelastic materials. Some numerical verification for this analytical solution is given in this report.

Keywords: interface corner, order of stress singularity, stress intensity factor, anisotropic elasticity, viscoelastic material

1. Stress Singularity of Interface Corner in Anisotropic Elastic Materials
The analytical closed-form solutions for the orders of stress singularity induced by anisotropic elastic interface corners have been obtained in our previous study [Hwu et al., 2003]. Since the strain energy is bounded only the singular orders within the range $0 < \text{Re}(\delta) < 1$ is
considered in our study. If $\delta_c$ stands for the most critical singular order, the terms associated with $\delta_c$ will dominate the stress behavior in the neighborhood of corner tip. As shown in eqs. (2.10a-2.10e) in [Hwu and Kuo, 2007], the most critical singular order $\delta_c$ may be one real simple root, or one real double root with two independent eigenfunctions, or one real triple root with three independent eigenfunctions, or a pair of complex conjugate roots, or the combination of one real simple root and a pair of complex conjugate roots, etc.. In other words, there are some cases in which only one or two eigenfunctions exist for the most critical singular order $\delta_c$. With this understanding, it is found that several combinations of interface corners will not possess full modes of stress intensity factors. Followings are some examples which have only one or two modes of stress intensity factors [Kuo and Hwu, 2010a].

**Example 1: Special wedges yielding only mode I stress intensity factor.**

Since it is difficult to prove analytically which kinds of wedge combinations will yield only one or two modes of stress intensity factors for the most critical singular order $\delta_c$, the observation is made through several different numerical calculations. When a single wedge is made up of *specially orthotropic materials* whose principal material axes are aligned with the nature body axes of the problem like Figure 1a with $\gamma = 0^\circ$ or $90^\circ$ and $\beta \neq 0^\circ$, it is found that many zero components occur in the matrix of eigenfunctions, $\Lambda$, which will lead two stress intensity factors $K_{II}$ and $K_{III}$ vanish. The same situation occurs for a bi-wedge composed of two unidirectional fiber-reinforced composites whose properties are the same but fiber orientations are opposite with respect to $x_1$-axis such as that shown in Figure 1b with $\beta \neq 0^\circ$.

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![Figure 1: Special wedges yielding only mode I stress intensity factor](image)

**Example 2: Special wedges yielding only mode I and II stress intensity factors.**

When a single wedge is made up of *generally orthotropic materials* whose principal material axes are *not* aligned with the nature body axes of the problem like Figure 2a with $\gamma \neq 0^\circ$ or $90^\circ$, and $\beta \neq 0^\circ$, or a bi-wedge composed of two dissimilar orthotropic materials such as Figure 2b or 2c with $\beta \neq 0^\circ$, their most critical singular order $\delta_c$ may be a real simple root or a simple and complex conjugate roots. Therefore, for the present example the zero components in the matrix $\Lambda$ will lead to $K_{III} = 0$ if we concern only the most critical singular order.
2. Stress Intensity Factors for the Lower Singular Orders

From the examples shown in Section 1, we see that if only the most critical singular order of stresses is considered, certain modes of stress intensity factors will vanish. However, in practical applications, the loss of certain modes of stress intensity factors does not mean that it will not fracture by that mode since the stresses associated with the next critical singular order may dominate the failure behaviour. With this understanding, sometimes it is necessary to compute the stress intensity factors associated with the lower singular orders. To provide a complementary definition for the stress intensity factors associated with the lower singular orders, the near tip solutions shown in eq. (2.1) in [Kuo and Hwu, 2010a] should be expanded to include all the possible singular orders, i.e.,

\[
\begin{align*}
    u(r, \theta) &= u_1(r, \theta) + u_2(r, \theta) + u_3(r, \theta) + ... , \\
    \phi(r, \theta) &= \phi_1(r, \theta) + \phi_2(r, \theta) + \phi_3(r, \theta) + ... .
\end{align*}
\]  

(1)

\( u_1, \phi_1 \) are the displacement vector and stress function vector associated with the most critical singular order while \( u_2, \phi_2 \) and \( u_3, \phi_3 \) are those associated with the next two critical singular orders, \( \delta_2 \) and \( \delta_3 \), which may be expressed as

\[
\begin{align*}
    u_j(r, \theta) &= \frac{1}{\sqrt{2\pi}} r^{1-\delta_j} V_j(\theta) < (1 - \delta_R^{(j)} + i \varepsilon_n^{(j)})^{-1} (r/\ell)^{i\varepsilon_R^{(j)}} > \Lambda_j^{-1} k_j, \\
    \phi_j(r, \theta) &= \frac{1}{\sqrt{2\pi}} r^{1-\delta_j} A_j(\theta) < (1 - \delta_R^{(j)} + i \varepsilon_n^{(j)})^{-1} (r/\ell)^{i\varepsilon_R^{(j)}} > \Lambda_j^{-1} k_j, \quad j = 2, 3, ....
\end{align*}
\]  

(2)

The subscript \( j \) or the superscript \( (j) \) denotes the value related to the \( j \)th critical singular order.
By the way similar to the stress intensity factors $k$ defined for the most critical singular order [NSC 95-2221-E-006-144-MY3], the stress intensity factors $k_j$, $j = 2, 3, \ldots$ associated with the $j$th critical singular order can be defined as

$$ k_j = \lim_{r \to 0} \sqrt{2\pi r} e^{i\phi_j(r)} A_j < (r/\ell)^{-i\phi_j(r)} > A_j^{-1} [\phi_j(r, 0) - \sum_{i=1}^{j-1} \phi_{i,r}(r, 0)], \quad j = 2, 3, \ldots \quad (3) $$

3. Mathematical Model of Anisotropic Viscoelasticity

In a fixed rectangular coordinate system $x_i$, $i = 1, 2, 3$, let $u_i(x, t)$, $\sigma_{ij}(x, t)$, and $\varepsilon_{ij}(x, t)$ be, respectively, the displacements, stresses, and strains. The constitutive law for the homogeneous anisotropic viscoelastic materials, the strain-displacement relation for the small deformation, and the equilibrium equation for steady state problems can be written as follows [Christenson, 1982]:

$$ \sigma_{ij}(x, t) = C_{ijkl}(t) \varepsilon_{kl}(x, 0) + \int_0^t C_{ijkl}(t - \tau) \frac{\partial \varepsilon_{kl}(x, \tau)}{\partial \tau} \, d\tau, \quad (4a) $$

$$ \varepsilon_{ij}(x, t) = \frac{1}{2} \left( \frac{\partial u_i(x, t)}{\partial x_j} + \frac{\partial u_j(x, t)}{\partial x_i} \right), \quad (4b) $$

$$ \frac{\partial \sigma_{ij}(x, t)}{\partial x_j} = 0, \quad (4c) $$

where $i, j, k, l = 1, 2, 3$; $x = (x_1, x_2, x_3)$ denotes spatial coordinates and $t$ denotes time; $C_{ijkl}(t)$ is the modulus determined through relaxation tests. If we define two different transformations for function $f(t)$ as

Laplace transformation: $\tilde{f}(s) = \int_0^\infty f(t)e^{-st} \, dt,$

Carson transformation: $\tilde{f}(s) = s\tilde{f}(s), \quad (5)$

and assume that the boundary of a body is invariant with time, the Laplace transformations of (4a-4c) give

$$ \tilde{\sigma}_{ij}(x, s) = \tilde{C}_{ijkl}(s) \tilde{\varepsilon}_{kl}(x, s), \quad (6a) $$

$$ \tilde{\varepsilon}_{ij}(x, s) = \frac{1}{2} \left( \frac{\partial \tilde{u}_i(x, s)}{\partial x_j} + \frac{\partial \tilde{u}_j(x, s)}{\partial x_i} \right), \quad (6b) $$

$$ \frac{\partial \tilde{\sigma}_{ij}(x, s)}{\partial x_j} = 0, \quad (6c) $$

in which $s$ is the transform variable. The mathematical forms of (6a-6c) are identical to the basic equations in linear elasticity. Consequently, the well-known correspondence principle between linear elasticity and linear viscoelasticity [Read, 1950; Sips, 1951; Brull, 1953; Lee, 1955] was established, that says: The viscoelastic solutions can be constructed directly from the solutions of the corresponding elastic problems by replacing the material properties with their Carson transformations, e.g. replacing $C_{ijkl}$ with $\tilde{C}_{ijkl}(s)$ or replacing $S_{ijkl}$ with $\tilde{S}_{ijkl}(s)$, while the stresses, strains, and displacements in elastic solutions should be replaced with their Laplace transformations for viscoelastic solutions.

By combining the correspondence principle and Stroh formalism, the general solutions satisfying the 15 basic equations in eq. (6) are

$$ \tilde{u}(z, s) = 2 \text{Re} \{ A(s) \tilde{f}(z, s) \}, \quad \tilde{\phi}(z, s) = 2 \text{Re} \{ B(s) \tilde{f}(z, s) \} \quad (7a) $$

and
where $\tilde{u}$ and $\tilde{\phi}$ are the Laplace transformations of displacement vector and stress function vector, respectively; $f(z, s)$ is a holomorphic function vector with complex variables $z_k, k = 1, 2, 3$; $\mu_k$ and $(a_k, b_k)$ are the material eigenvalues and eigenvectors in Laplace-domain ($s$-domain) and can be determined by the Stroh sextic eigenrelation of anisotropic elasticity [Stroh, 1958; Ting, 1996], in which the fundamental elasticity matrix $N$ is now a function of $s$ whose based matrices $Q, R,$ and $T$ are three $3 \times 3$ real matrices defined by $\tilde{C}_{ijkl}(s)$ instead of $C_{ijkl}(t)$.

By taking Laplace inversion of (7a), the general solutions of displacements and stresses in time-domain ($t$-domain) can be obtained as

$$u(z, t) = 2 \Re\{A_i(t) * f_i(z, t)\}, \quad \phi(z, t) = 2 \Re\{B_i(t) * f_i(z, t)\},$$

(8)

where the transformation relations between $A & A_i$, $B & B_j$, and $f & f_i$ are unclear yet and will be studied in the next year; the operator $*$ denotes the Stieltjes convolution, that is

$$\varphi(t) * \psi(t) = \int_{-\infty}^{\infty} \varphi(t - \tau) \frac{\partial \psi(\tau)}{\partial \tau} d\tau.$$

(9)

Note that the stress functions, $i = 1, 2, 3$, are related to the stresses $\sigma_{ij}$ by

$$\sigma_{\theta\theta} = -n^T \frac{\partial \tilde{\phi}(t)}{\partial x_2}, \quad \sigma_{rr} = -s^T \frac{\partial \tilde{\phi}(t)}{\partial x_1}, \quad \sigma_{\theta\theta} = s^T \frac{\partial \tilde{\phi}(t)}{\partial r} = -n^T \frac{\partial \tilde{\phi}(t)}{\partial \theta},$$

(10a)

$$\sigma_{\theta\theta} = n^T \frac{\partial \tilde{\phi}(t)}{\partial r} \quad \sigma_{r\theta} = -s^T \frac{\partial \tilde{\phi}(t)}{\partial \theta}, \quad \sigma_{r\theta} = s^T \frac{\partial \tilde{\phi}(t)}{\partial r} = -n^T \frac{\partial \tilde{\phi}(t)}{\partial \theta},$$

(10b)

where $s^T$, $n^T$, and $t^T_3$ are

$$s^T = (\cos \theta, \sin \theta, 0), \quad n^T = (-\sin \theta, \cos \theta, 0), \quad t^T_3 = (0, 0, 1),$$

(10c)

in which $\theta$ is the angle between $x_1$-axis and tangent direction, and the superscript $T$ denotes the transpose of a vector or matrix.

It seems impossible to invert all kinds of functions by using the analytical Laplace inversion, particularly for complicated functions. Therefore, in our follow-up study numerical Laplace inversion will be considered, e.g. Schapery’s collocation method [Schapery, 1962] and Stehfest method [Stehfest, 1970].

4. Stress Singularity of Interface Corner in Viscoelastic Materials

The boundary conditions for the interface corners in anisotropic viscoelastic materials can be expressed in Laplace domain as

$$\tilde{u}(0, s) = \tilde{u}_2(0, s), \quad \tilde{\phi}(0, s) = \tilde{\phi}_2(0, s), \quad \tilde{\phi}(\theta_0, s) = \tilde{\phi}_2(\theta_2, s) = 0.$$  

(11)

Since the mathematical form of the basic equations (6), the general solutions (7), and the boundary conditions (11) are totally the same as those of the corresponding problem of anisotropic elastic materials, the solution for the viscoelastic interface corners should also
preserve the same mathematical form as the solution for elastic interface corners when the correspondence principle is employed. With this understanding, the analytical closed-form solutions of stress singularity obtained in [Hwu et al., 2003] are also applicable to the viscoelastic materials. One should note that the matrix $K^{(3)}_v$ in this closed-form solution and the singular order $\delta$ determined from $\|K^{(3)}_v\| = 0$ are both functions of the transform variable $s$. Since the function form of $K^{(3)}_v(s)$ is so complicated, the singular order $\delta(s)$ in Laplace domain cannot be easily solved even when the symbolic-computation software, e.g. Mathematica and Matlab, is employed. Here, another way is adopted to calculate the order of stress singularity in $t$-domain, i.e., the Schapery’s collocation method [Schapery, 1962]. By defining the value of singular order in $t$-domain as $\ln(\delta_i(t))$, we can evaluate it via

$$\ln(\delta_i(t)) = \ln(\delta_i(\infty)) + \sum_{j=1}^n b_i e^{-d_j t},$$

where $\ln(\delta_i(t))$ is the approximation of $\ln(\delta_i(t))$; $\ln(\delta_i(\infty))$ is the value of $\ln(\delta_i(t))$ as its argument $t$ approaches to infinity; $n$ is a natural number, $d_j$ are the prescribed positive constants; $b_i$ can be determined via $n$ simultaneous equations as

$$\ln(\delta(s)) - \ln(\delta_i(\infty)) \approx \sum_{j=1}^n \frac{b_i}{s + d_j},$$

Equation (12) means that we can select $n$ values of $d_j$ and substitute these into the equation $\|K^{(3)}_v\| = 0$. Then, we can solve the corresponding singular orders $\ln(\delta(s))_{s \to -d_j}$ and use these results to construct the approximation of $\ln(\delta_i(t))$ through eq. (12).

In order to verify the correctness of the proposed approach, several different cases have been done in this project. To save the space of this report, here we only present the results for an interface crack between an orthotropic viscoelastic material and an orthotropic elastic material. Figure 3 shows that the time response of the oscillatory index $\ln(\delta_i(t))$ calculated by the proposed approach agree excellently with the analytical solution.

![Figure 3: Time variation of the oscillatory index $\ln(\delta_i(t))$ for an interface crack between an orthotropic viscoelastic material and an orthotropic elastic material.](image)

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References