壓電材料之進階力學問題 (2/3)
Some Advanced Mechanics Problems of Piezoelectric Materials (2/3)

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Abstract

In the first year of this project, four different types of problems were solved: (1) mathematical model for piezoelectric anisotropic elasticity; (2) singular orders of piezoelectric multiwedges; (3) contact problems of piezoelectric materials; (4) coupled stretching-bending analysis for piezoelectric laminates. Following the results of the first year, in this year another four problems were done. They are: (1) fracture parameters for piezoelectric materials; (2) stress intensity factors of piezoelectric multiwedges; (3) BEM design for contact problems of piezoelectric materials; (4) BEM design for piezoelectric laminates. Most of the works have been completed during this year.

Keywords: piezoelectric materials, anisotropic elasticity, contact problem, piezoelectric multiwedges, electromechanical analysis, boundary element analysis, stretching-bending coupling analysis, multi-scaled finite element analysis, fracture parameters, corners/interface cracks, molecular dynamic simulation, nanoindentation

1. Fracture Parameters for Piezoelectric Materials

After setting up the mathematical model for piezoelectric anisotropic elasticity and obtaining the explicit expressions for the important matrices involved in the extended Stroh formalism (Hwu, 2008), the main task of this year is exploring the appropriate definition for the fracture parameters of piezoelectric materials, especially for the cases of corner problems.

A proper definition for the interface corner has been proposed recently (Hwu and Kuo, 2007), which can be reduced to the conventional definition for cracks in homogeneous anisotropic materials or for cracks along the interfaces between two dissimilar anisotropic materials. Based upon the definition proposed for the interface corners of anisotropic media, in this project we extend this definition to the general cases of piezoelectric media, which can now be written as

\[
\begin{bmatrix}
K_{II} \\
K_{I} \\
K_{III} \\
K_{IV}
\end{bmatrix}
= \lim_{r \to 0, \theta \to 0} \sqrt{2\pi r}^{\delta_{s}} \Lambda \left( r / \ell \right)^{-\epsilon_{s}} > \Lambda^{-1} \begin{bmatrix}
\sigma_{\theta\theta} \\
\sigma_{\theta\phi} \\
\sigma_{\phi\phi} \\
D_{\theta}
\end{bmatrix},
\]

(1a)

or in matrix form

\[
k = \lim_{r \to 0, \theta \to 0} \sqrt{2\pi r}^{\delta_{s}} \Lambda \left( r / \ell \right)^{-\epsilon_{s}} > \Lambda^{-1} \phi_{r}(r, \theta).
\]

(1b)

In the above, \(K_I, K_{II}, K_{III}\) and \(K_{IV}\) are the opening, shearing, tearing and electric stress...
intensity factors; \((\sigma_{r\theta}, \sigma_{\theta\theta}, \sigma_{rr})\) and \(D_\theta\) are, respectively, the stresses and electric displacement in polar coordinate \((r, \theta)\) whose origin is located at the corner tip; \(\delta_k\) and \(\varepsilon_a, \alpha = 1, 2, 3, 4\) are the real part and imaginary part of the most critical singular order; \(\ell\) is a reference length employed to smooth the physical unit of \(k\) associated with the complex singular orders; \(\Lambda\) is a matrix related to the eigenfunctions of the displacements, stresses, electric fields and electric displacements near the apex of piezoelectric interface corners; \(\phi\) is the stress function vector and the subscript comma denotes the differentiation.

The near tip solution can then be written in terms of the stress/electric intensity factors \(k\) as (Hwu and Kuo, 2010)

\[
\begin{align*}
\mathbf{u}(r, \theta) &= \frac{1}{\sqrt{2\pi}} r^{3-\delta_k} \mathbf{V}(\theta) < (1 - \delta_k + i\varepsilon_a)^{-1}(r / \ell)^{i\varepsilon_a} > \Lambda^{-1} \mathbf{k}, \\
\phi(r, \theta) &= \frac{1}{\sqrt{2\pi}} r^{3-\delta_k} \mathbf{\Lambda}(\theta) < (1 - \delta_k + i\varepsilon_a)^{-1}(r / \ell)^{i\varepsilon_a} > \Lambda^{-1} \mathbf{k}, \\
\phi_j(r, \theta) &= \frac{1}{\sqrt{2\pi}} r^{3-\delta_k} \mathbf{\Lambda}(\theta) < (1 - \delta_k + i\varepsilon_a)^{-1}(r / \ell)^{i\varepsilon_a} > \Lambda^{-1} \mathbf{k},
\end{align*}
\]

(2)

For the problems of cracks or interface cracks or interface corners, it is quite apparent to choose \(\theta = 0\) along the crack or interface to study the intensity factors of stresses on the surface parallel to the crack or interface. For the general corner problems, if we choose \(\theta = 0\) on different directions, different stress intensity factors are studied. In our proposal, we thought that like the stresses are second order tensors, the stress intensity factors for different directions may also be second order tensors. However, in our study we found that it is not true but the concept of the principal stress intensity factor is really useful for the prediction of propagation of corner tip, which will then be studied in our future work.

In our study we also found that if only the most critical singular order is considered, certain modes of stress intensity factors will be lost even though the specimen is subjected to mixed-mode loading. The missing modes may come from the stress/electric fields associated with the lower singular orders. Hence, to provide a proper definition for the stress/electric intensity factors associated with the lower singular orders, the near tip solutions (2) should be extended to include all the possible singular orders, i.e.,

\[
\begin{align*}
\mathbf{u}(r, \theta) &= \mathbf{u}_1(r, \theta) + \mathbf{u}_2(r, \theta) + \mathbf{u}_3(r, \theta) + \ldots, \\
\phi(r, \theta) &= \phi_1(r, \theta) + \phi_2(r, \theta) + \phi_3(r, \theta) + \ldots,
\end{align*}
\]

(3)

where \(\mathbf{u}_1\) and \(\phi_1\) are generalized displacement vector and generalized stress function vector associated with the most critical singular order, which are shown in (2), and \(\mathbf{u}_2, \phi_2\) and \(\mathbf{u}_3, \phi_3\) are generalized displacement vectors and generalized stress function vectors associated with the next two critical singular orders, which may be expressed as

\[
\begin{align*}
\mathbf{u}_j(r, \theta) &= \frac{1}{\sqrt{2\pi}} r^{3-\delta_j} \mathbf{V}_j(\theta) < (1 - \delta_k^{(j)} + i\varepsilon_a^{(j)})^{-1}(r / \ell)^{i\varepsilon_a^{(j)}} > \Lambda_j^{-1} \mathbf{k}_j, \\
\phi_j(r, \theta) &= \frac{1}{\sqrt{2\pi}} r^{3-\delta_j} \mathbf{\Lambda}_j(\theta) < (1 - \delta_k^{(j)} + i\varepsilon_a^{(j)})^{-1}(r / \ell)^{i\varepsilon_a^{(j)}} > \Lambda_j^{-1} \mathbf{k}_j, \quad j = 2, 3, \ldots,
\end{align*}
\]

(4)

where the subscript \(j\) or the superscript \((j)\) denotes the value associated with the \(j\)th critical singular order \(\delta^{(j)} = \delta_k^{(j)} + i\varepsilon_a^{(j)}, \alpha = 1, 2, 3, 4\). By the way similar to \(k\), the stress/electric intensity factors \(k_j\) can be defined as
After finding the proper definition for the stress intensity factors of general corners of piezoelectric materials, the singular characteristics of cracks or interface cracks in piezoelectric materials can be studied through the singular orders of stresses, the near tip solutions and the relations between crack opening displacement, strain energy density, and stress intensity factors (Hwu and Ikeda, 2008).

2. Stress Intensity Factors of Piezoelectric Multiwedges

To provide a stable and efficient computing approach for the general mixed-mode stress intensity factors, similar to the anisotropic multiwedges studied in our previous project, the path-independent H-integral based on reciprocal theorem of Betti and Rayleigh is also adopted in the present study, which is

$$ H = \int_{\Gamma} (\mathbf{u}^{T} \hat{\mathbf{i}} - \hat{\mathbf{i}}^{T} \mathbf{t}) d\Gamma, $$

where $\Gamma$ is a counterclockwise integral path with arbitrary shape which emanates from the lower corner flank ($\theta = \theta_0$) and terminates at the upper corner flank ($\theta = \theta_n$); $\mathbf{u}$ and $\mathbf{t}$ are the generalized displacement and traction vectors of the actual system which can be obtained through any method, e.g. finite element analyses, boundary element analyses, or even experiment measurements, while $\hat{\mathbf{u}}$ and $\hat{\mathbf{i}}$ are those of the auxiliary system with singular order $2 - \delta$. Expressed in component form, the H-integral (6) can be written as

$$ H = \int_{\Gamma} (u_1 \hat{t}_1 + u_2 \hat{t}_2 + u_3 \hat{t}_3 + u_4 \hat{t}_4 - \hat{u}_1 t_1 + \hat{u}_2 t_2 + \hat{u}_3 t_3 + \hat{u}_4 t_4) d\Gamma, $$

where

$$ t_i = \sigma_{ij} n_j + \sigma_{in} n_i, \quad i = 1, 2, 3, \quad t_4 = D_1 n_1 + D_2 n_2, $$

$$ \hat{t}_i = \hat{\sigma}_{ij} n_j + \hat{\sigma}_{in} n_i, \quad i = 1, 2, 3, \quad \hat{t}_4 = \hat{D}_1 n_1 + \hat{D}_2 n_2, $$

in which $(n_1, n_2)$ is the normal direction of integration path $\Gamma$ and $n_3 = 0$ due to the 2D feature. If the integral path $\Gamma$ is selected to be a circular path, we may choose the polar coordinate to represent the displacements and stresses, and equation (7a) can be further reduced to

$$ H = \int_{\theta_0}^{\theta} (u_1 \hat{\sigma}_{1\theta} + u_2 \hat{\sigma}_{2\theta} + u_3 \hat{\sigma}_{3\theta} - \hat{u}_1 \sigma_{1\theta} - \hat{u}_2 \sigma_{2\theta} - \hat{u}_3 \sigma_{3\theta}) r dr. $$

If the integral path $\Gamma$ is selected to be a rectangular path, equation (7a) can be further reduced to

$$ H = \int (u_1 \hat{\sigma}_{12} + u_2 \hat{\sigma}_{22} + u_4 \hat{D}_2 - \hat{u}_1 \sigma_{12} - \hat{u}_2 \sigma_{22} - \hat{u}_4 D_2) dx + $$

$$ \int (u_1 \hat{\sigma}_{11} + u_2 \hat{\sigma}_{22} + u_4 \hat{D}_1 - \hat{u}_1 \sigma_{11} - \hat{u}_2 \sigma_{22} - \hat{u}_4 D_1) dy. $$

Note that equations (8) and (9) are reduced for the general two-dimensional cases whose $u_3 = 0$ or $t_3 = 0$.

A direct relation between H-integral and stress/electric intensity factors associated with the most critical singular order $\delta$ has also been obtained as (Hwu and Kuo, 2010)

$$ k = \sqrt{2\pi} \Lambda < (1 - \delta + i \varepsilon_a) e^{i\varepsilon_a} > H^* h, $$

where

$$ H^* = \int_{\theta_0}^{\theta} (\hat{\Lambda}^T (\theta) V(\theta) - \hat{V}^T (\theta) \Lambda(\theta)) d\theta, \quad h = \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{bmatrix}, $$

$$ \Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \\ \Lambda_4 \end{bmatrix}, \quad \Lambda_1 = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}. $$

$$ \Lambda_2 = \begin{bmatrix} \Lambda_{21} & \Lambda_{22} \\ \Lambda_{31} & \Lambda_{32} \end{bmatrix}, \quad \Lambda_3 = \begin{bmatrix} \Lambda_{31} & \Lambda_{32} \\ \Lambda_{41} & \Lambda_{42} \end{bmatrix}, \quad \Lambda_4 = \begin{bmatrix} \Lambda_{41} & \Lambda_{42} \\ \Lambda_{51} & \Lambda_{52} \end{bmatrix}. $$
and $H_i$, $i=1,2,3,4$, be the value of $H$ with $\hat{c}_i=1$ and $\hat{c}_j=0$, $i \neq j$, and $\hat{c}_i$ are the coefficients of the complimentary solutions (see (Hwu and Kuo, 2010) for the detailed explanation of all the symbols used in (10b)).

Figure 1 is a plot showing the effects of electric load on the stress/electric intensity factors of interface corners between two dissimilar piezoelectric materials (see (Hwu and Kuo, 2010) for the detailed explanation of this example).

![Figure 1. Stress/electric intensity factors versus electric load $D_0$ for interface corner $\beta = 30^\circ$ between PZT-5H and PZT-7A](image)

3. **BEM Design for Contact Problems of Piezoelectric Materials**

Finite element modeling (FEM) and boundary element modeling (BEM) are two important tools for stress analysis of structures. With known structural arrangements, mechanical properties and loading environments, the structural responses such as deformations and stresses can be observed through these two popular tools. By constructing the training patterns connecting mechanical properties and structural responses of a certain specimen, the mechanical properties of unknown materials can be characterized through some necessary measured data and their associated well-trained neural network. This kind of techniques is commonly used in the structural materials with ordinary sizes. When structure sizes move to the micro- or even nano-scales, the validity of numerical modeling becomes an important issue which may influences the characterization of micro- or nano-materials. It's known that for some specific problems BEM is much more efficient and accurate than FEM if its associated fundamental solution can be found analytically. With this understanding, in this topic we develop a special BEM that may be helpful for the characterization of micro- or nano-materials. One of the tools that are widely used for such measurement is depth-sensing indentation.

To develop an accurate and efficient BEM dealing with indentation problems, in this study
the Green’s function of anisotropic bimaterials is used as the fundamental solutions of BEM. Through this BEM, the stress analysis of indentation on layered materials can be performed more accurate and efficient. In this study, the corresponding Green’s functions of piezoelectric bi-materials have been obtained and employed to construct the BEM for the contact problems of piezoelectric materials. They are (refer to (Chen and Hwu, 2010) for the detailed explanation of the symbols and related discussions)

\[
\begin{align*}
\mathbf{u}_1 &= 2 \text{Re} \{ \mathbf{A}_1 [f_0(z^{(1)}) + f_1(z^{(1)})] \}, \\
\mathbf{u}_2 &= 2 \text{Re} \{ \mathbf{A}_2 f_2(z^{(2)}) \}, \\
\mathbf{\phi}_1 &= 2 \text{Re} \{ \mathbf{B}_1 [f_0(z^{(1)}) + f_1(z^{(1)})] \}, \\
\mathbf{\phi}_2 &= 2 \text{Re} \{ \mathbf{B}_2 f_2(z^{(2)}) \},
\end{align*}
\]

where the subscripts 1 and 2 or the superscripts (1) and (2) denote materials 1 and 2, respectively.

\[
\begin{align*}
f_0(z^{(1)}) &= \frac{1}{2\pi} < \ln(z_a - \bar{z}_a) > \mathbf{A}_1^T \hat{\mathbf{p}}, \\
f_1(z^{(1)}) &= \frac{1}{2\pi} \sum_{j=1}^{3} < \ln(z_a - \bar{z}_j) > \mathbf{A}_1^{-1} (\mathbf{M}_2 + \mathbf{M}_1)^{-1} (\mathbf{M}_2 - \mathbf{M}_1) \mathbf{A}_j \mathbf{A}_1^T \hat{\mathbf{p}}, \\
f_2(z^{(2)}) &= -\frac{1}{2\pi} \sum_{j=1}^{3} < \ln(z_a - \bar{z}_j) > \mathbf{A}_2^{-1} (\mathbf{M}_2 + \mathbf{M}_1)^{-1} \mathbf{A}_1^T \mathbf{A}_j \mathbf{A}_1^T \hat{\mathbf{p}},
\end{align*}
\]

in which \( \mathbf{M}_j, j = 1,2, \) is the impedance matrix defined by

\[
\mathbf{M}_j = -i \mathbf{B}_j \mathbf{A}_j^{-1}, j = 1,2.
\]

Figure 2 is a mesh diagram of the BEM on contact problems, which shows that the main difference between the present BEM and the traditional BEM is the meshes on the interface. In the present BEM, more accurate results are obtained by relatively coarse meshes (no meshes on the interface). Detailed numerical examples for the general piezoelectric bi-materials can be found in (Chen and Hwu, 2010).
4. BEM Design for Piezoelectric Laminates

There are two important works on BEM design. One is the derivation of the fundamental solution, which is sometimes called Green’s function, and the other is the formulation of boundary integral equations. Besides these two works, there are still many other technical details needed to be worked out such as singular integration, free term coefficients, the treatment of domain integrals and corner forces as well as the position setting of boundary source points, etc. In the first year of this project, the main task is the finding of the Green’s functions for piezoelectric materials, which has been done and the results are published in (Chen and Hwu, 2010). While this year, our main attention was focused on the establishment of the boundary integral equations for the stretching-bending coupling analysis of composite laminates. We start from the reciprocal theorem of Betti and Raleigh as that shown in eqn.(6). Starting from this theorem, the boundary integral equations for symmetric/unsymmetric laminates were derived in our previous project (NSC 95-2221-E-006-240) and should be revised when we tried to implement these boundary integral equations on BEM in this project. Following is the revised version of the boundary integral equations

\[
\begin{align*}
    c_{ij}(\xi) u_{ij}(\xi) &+ J_{ij}(\xi) + \int_{r_x} t_{ij}^*(\xi, x) u_{ij}(x) d\Gamma(x) + \sum_{k=1}^{N_r} \tilde{t}_{ij}^*(\xi, x_k) \tilde{u}_{ij}(x_k) \\
    & = \int_{r_x} u_{ij}^*(\xi, x) v_j(x) d\Gamma(x) + \int_{A} u_{ij}^*(\xi, x) q_j(x) dA(x) + \sum_{k=1}^{N_r} u_{ij}^*(\xi, x_k) t_{ij}(x_k),
\end{align*}
\]

where

\[
\begin{align*}
    \tilde{u}_{ij}(x) & = u_{ij}(x), \quad j = 1, 2, 4, \\
    \tilde{u}_{ij}(x) & = u_{ij}(x) - u_{ij}(\xi)
\end{align*}
\]

and

\[
\begin{align*}
    c_{ij}(\xi) u_{ij}(\xi) & = \lim_{\varepsilon \to 0} \left\{ \int_{r_x} t_{ij}^*(\xi, x) \tilde{u}_{ij}(x) d\Gamma(x) \right\} \\
    J_{ij}(\xi) & = \lim_{\varepsilon \to 0} \left\{ \int_{r_x} t_{ij}^*(\xi, x) \tilde{u}_{ij}(x) d\Gamma(x) \right\} - \int_{r_{\varepsilon}} u_{ij}^*(\xi, x) v_j(x) d\Gamma(x)
\end{align*}
\]

\[
\begin{align*}
    u_1 = u_0, \quad u_2 = v_0, \quad u_3 = w, \quad u_4 = \beta_n, \\
    t_1 = T_x = N_{x}n_x + N_{y}n_y, \quad t_2 = T_y = N_{x}n_x + N_{y}n_y, \quad t_3 = V_n, \quad t_4 = M_n, \\
    q_1 = q_x, \quad q_2 = q_y, \quad q_3 = q, \quad q_4 = m_n, \\
    t_c = M_{ns}^+ - M_{ns}^-
\end{align*}
\]

\[
\begin{align*}
    u_{ij}^*(\xi, x), \quad t_{ij}^*(\xi, x) \quad \text{and} \quad t_c^*(\xi, x) \quad \text{represent, respectively}, \quad u_{ij}, t_{ij} \quad \text{and} \quad t_{ij} \quad \text{at point} \quad x \quad \text{corresponding} \quad \text{to a unit point force acting in the} \quad e_i \quad \text{direction applied at point} \quad \xi.
\end{align*}
\]

These four equations in terms of four unknown functions \( u_{ij} \) or \( t_{ij}, j = 1, 2, 3, 4 \), constitute the basis of the boundary element formulation. With this revision, currently we tried to extend these boundary integral equations to the piezoelectric laminates and then design BEM by following the standard procedures stated in most of the textbooks on BEM such as (Brebbia, et al., 1984).
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References