The Analogy Between the Interface Crack Problems and the Punch Problems

Abstract
At the first glance, it is not easy to see any connection between the punch problems and the interface crack problems. However, the similarity such as, the stress oscillatory singularity characteristics near the interface crack tips or the punch corners, stimulates us to find the connection between these two problems. If we can find the analogy between them, many known punch problems can be solved through the interface crack problems, or vice versa. In the mechanics community, solving problems by analogy technology is not unusual. The well-known analogous examples are forces and dislocations, cracks and rigid line inclusions, and holes and rigid inclusion. In this report, we try to re-express the boundary conditions for these two problems and rewrite their corresponding solutions into an analogous form. From the reorganized expressions, we see that if the material above the interface is rigid the solution forms of these two problems can be made to be equivalent by only interchanging the material eigenvectors A and B. To get a complete analogous solution, all the boundary conditions including the conditions of the outer boundary should be analogous to each other. A traction-free interface crack problem and a flat-ended punch problem are solved completely in a similar way to illustrate the analogy. Moreover, some representative punch problems are solved completely, and the real form solutions for the contact pressure and surface deformations of these problems are also derived. Their related stresses contours, surface deformations of contact pressures are also plotted to help us see more clearly the physical behaviors of the punch problems.

Keywords: Interface Crack Problem, Punch Problem, Anisotropic Elasticity, Analogous Relation

1. Introduction
Both of the interface crack problems and the punch problems belong to the mixed type boundary value problems. The outlooks of the boundary conditions of these two problems are very different. The former states the displacement and traction continuity across the uncracked portions, and the traction-prescribed conditions along the cracked portions. The latter (if the punch is considered to be rigid) states the displacement-prescribed condition along the contact regions, and the traction-free condition along the uncontact regions. Because of this difference, these two problems are usually solved independently (Muskhelishvili, 1954; Hwu, 1993a; Fan and Hwu, 1996). However, their solutions show some similarities such as the stress oscillatory singularity characteristics near the interface crack tips or the punch corners. This similarity stimulates us to find the connection between these two problems. By carefully reviewing these two different boundary conditions, we find that the punch problem is just a counterpart of the interface crack problem with one of the materials to be rigid. Hence, similar to the analogy between forces and dislocations, cracks and rigid line inclusions, or holes and rigid inclusion, we may now solve the punch problems by analogy with the interface crack problems, or vice versa. This finding is useful not only in analysis but also in experiment. Because one may understand the physical behavior of the interface crack by doing the experiment of punch problems, or vice versa.

The analogy between the contact problems and the crack problems has been noticed before, e.g. (Willis, 1968; Brock, 1978). However, to the
authors’ knowledge, these two kinds of problems were solved independently and no detailed analogous relations have been provided. In this report, the analogy will be discussed for the most general two-dimensional anisotropic linear elastic materials. Through this study, it is hoped that the analogy can be extended to the three-dimensional contact and crack problems since both of these two problems were usually formulated as problems of Boussinesq type (Sneddon, 1946, 1969; Willis, 1966, 1967, 1968, 1970; Gladwell, 1980).

2. Problem Statements and Solutions

2.1 Governing Equation

The basic equations for linear anisotropic elasticity are the strain-displacement equations, the stress-strain laws and the equations of equilibrium. For two-dimensional problems in which \( x_3 \) does not appear in the basic equations and the boundary conditions, the general solution to these basic equations may be expressed in terms of three holomorphic functions of complex variables (Stroh, 1958; Lekhnitskii, 1963). This enables us to apply many of the powerful results of complex function theory to the two-dimensional elasticity. For the later use of derivation, we now list a compact matrix form solution (Stroh, 1958; Ting, 1986) which satisfies all the basic equations, i.e.,

\[
\begin{bmatrix}
0 \\
\text{Re}\{\phi^T(z)\} \\
\text{Re}\{\mathbf{f}(z)\}
\end{bmatrix}
\]

(1a)

where

\[
\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], \quad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3],
\]

\[
\mathbf{f}(z) = [f_1(z_1), f_2(z_2), f_3(z_3)]^T,
\]

\[
z_\alpha = x + p_{\alpha} y, \quad \alpha = 1, 2, 3.
\]

(1b)

In the above equations, \( \mathbf{u} = (u_1, u_2, u_3) \) is the vector form of displacement, \( \phi = (\phi_1, \phi_2, \phi_3) \) stands for the stress function vector which is related to the stresses \( \sigma_{ij} \) and surface traction \( \mathbf{t} \) by

\[
\sigma_{ij} = -\phi_{ij}, \quad \mathbf{t} = \frac{\partial \phi}{\partial s},
\]

(1c)

and

\[
\mathbf{t} = \mathbf{f}.
\]

(1d)

where \( s \) is the arc length measured along the curved boundary, \( p_\alpha, \alpha = 1, 2, 3, \) are the material eigenvalues whose imaginary parts have been arranged to be positive; \( (\mathbf{a}_\alpha, \mathbf{b}_\alpha) \) are their associated eigenvectors. \( f_\alpha(z_\alpha) \) are three holomorphic complex functions to be determined by satisfying the boundary conditions. The superscript \( T \) denotes the transpose.

2.2 Boundary Conditions

\textbf{Interface Crack Problems}

Consider a set of cracks \( L \) lying along the interface of two dissimilar anisotropic materials. The materials are assumed to be perfectly bonded at all points of the interface except those lying in the region of cracks. To describe the boundary conditions of this kind of interface crack problems, we need to consider the displacement and traction continuity across the uncracked portions, and the traction-prescribed conditions along the cracked portions. Thus,

\[
\phi'_{\alpha}(x) = \phi_{\alpha}(x) = \mathbf{i}, \quad x \in L
\]

\[
\mathbf{u}_1(x) = \mathbf{u}_2(x), \quad \phi_1(x) = \phi_2(x), \quad x \not\in L
\]

(2)

where \( \mathbf{i} \) is the prescribed traction along the crack surface; prime denotes differentiation with respect to its argument. The symbols marked with the subscripts 1 and 2 represents, respectively, the quantities pertaining to the materials located upper and lower the interface. If we consider the material above the interface to be rigid, the boundary conditions (2) will then be specialized to

\[
\phi'(x) = \mathbf{i}, \quad x \in L
\]

\[
\mathbf{u}(x) = \mathbf{0}, \quad x \not\in L
\]

(3)

Note that in eqn.(3) and the following derivation the subscript 2 is dropped for the convenience of presentation, and the subscript 1 will not enter into the boundary conditions since material 1 is assumed to be rigid.

Substituting (1a) into (3), we have

\[
\text{Re}\{\mathbf{f}(x)\} = \mathbf{i}, \quad x \in L,
\]

\[
\text{Re}\{\mathbf{A}(x)\} = \mathbf{0}, \quad x \not\in L.
\]

(4)

\textbf{Punch Problems}

Consider the case that a set of rigid punches \( L \) of given profiles are brought into contact with the surface of the half-plane and are allowed to indent the surface in such a way that the punches completely adhere to the half-plane on initial contact and during the subsequent indentation no slip occurs and the contact region does not change. The boundary conditions of this kind of punch problems may be expressed by the displacement-prescribed condition along the contact regions, and the traction-free condition along the uncontact regions. Hence,

\[
\mathbf{u}(x) = \mathbf{u}, \quad x \in L
\]

\[
\phi'(x) = \mathbf{0}, \quad x \not\in L
\]

(5)

Substituting (1a) into (5), we have

\[
\text{Re}\{\mathbf{A}(x)\} = \mathbf{u}, \quad x \in L,
\]

\[
\text{Re}\{\mathbf{f}(x)\} = \mathbf{0}, \quad x \not\in L.
\]

(6)

By comparison between eqns.(4) and (6), we see
that they are counterpart of each other. Therefore, we may deal with any one of the problems by analogy with the other problem.

### 2.3 The Analogy

One of the special features of the Stroh's formalism is that the solution form, eqn.(1), is neat and elegant. Due to its elegance, many important characteristics can be found at the first glance of the solution form. For example, the displacements and stress functions shown in eqn.(1a) are distinguished only by the material eigenvector matrices \( \mathbf{A} \) and \( \mathbf{B} \). Thus, the relevant boundary conditions of the displacement prescribed problems differ from those of the traction prescribed problems only in the appearance of the symbols \( \mathbf{A} \) and \( \mathbf{B} \). Since the mathematical formulations for the displacement prescribed problems and the traction prescribed problems are identical, their solutions should also be identical with \( \mathbf{A} \) and \( \mathbf{B} \) interchanged.

Both of the interface crack problems and the punch problems belong to the mixed type boundary value problems. Although the outlook of the boundary conditions of these two problems shown in (2) and (5) are very different, eqn.(3) (which is a special case of (2)) and eqn.(5) are almost identical. In order to see more clearly about their equivalency, we may differentiate the second equation of (4) and the first equation of (6) with respect to \( x \). The solutions \( \mathbf{f}(z) \) to these two problems should therefore be identical with \( \mathbf{A} \) and \( \mathbf{B} \) interchanged, which has been verified by using the solutions obtained in the literature (Hwu, 1993a; Fan and Hwu, 1996).

In addition to the analogy to the solutions of function \( \mathbf{f}(z) \), we now like to present some simple results for the analogy between the surface traction and deformation. It is known that the stress function vector \( \mathbf{t} \) and the displacement vector \( \mathbf{u} \) have the following relation (Yeh, et al., 1993a,b)

\[
\mathbf{A}^T \mathbf{t} + \mathbf{B}^T \mathbf{u} = \mathbf{f}(z) \tag{7}
\]

By this relation, many physical quantities can be obtained easily. For the interface crack problems, crack opening displacement \( \mathbf{u} \) along the crack surface \( x \in L \) can be found by substituting (4) into (7), and the stress distribution \( \mathbf{t} \) along the interface \( x \notin L \) can be found by substituting (4) into (7). For the punch problems, the contact pressure \( \mathbf{t} \) under the punch \( x \in L \) can be found by substituting (6) into (7), and the surface deformation \( \mathbf{u} \) outside the punch \( x \notin L \) can be found by substituting (6) into (7). The results are

**interface crack problem:** (the material above the interface is rigid)

\[
\begin{align*}
\mathbf{u}'(x) &= \mathbf{B}^T[\mathbf{f}(x) - \mathbf{A}^T \mathbf{t}(x)], \ x \in L \\
\mathbf{t}(x) &= \mathbf{A}^{-T} \mathbf{f}'(x), \ x \notin L
\end{align*} \tag{8}
\]

**punch problem:** (the punch is rigid)

\[
\begin{align*}
\mathbf{t}(x) &= \mathbf{A}^{-T} [\mathbf{f}(x) - \mathbf{B}^T \mathbf{u}(x)], \ x \in L \\
\mathbf{u}'(x) &= \mathbf{B}^{-T} \mathbf{f}'(x), \ x \notin L
\end{align*} \tag{9}
\]

The results obtained in (8) and (9) show that the surface traction and the displacement gradient are analogous to each other for these two different problems.

### 3. Representative Examples

For a real problem, the boundary conditions given in (4) and (6) are not complete since they only state the conditions along the interface or the half-plane surface. For a finite body, there should be a condition describing the outer boundary. For an infinite body, the outer boundary condition is the so-called infinity condition. Therefore, without knowing the conditions for the outer boundary (or infinity), the solutions obtained are incomplete. If the outer boundary condition (or infinity condition) does not possess the analogous characteristics like those shown in (4) and (6), the solutions may not have the analogous form for the interface crack problems and the punch problems. In order to have a better understanding about the analogy, we now choose the examples whose \( \mathbf{f} \) (or \( \mathbf{u} \)) is zero along the crack surface (or the contact region).

**A traction-free interface crack:** (the material above the interface is rigid)

Consider a finite interface traction-free crack located on \((-a, a)\) subjected to a uniform loading \( t^x \) at infinity. By using the general solutions obtained in the literature (Hwu, 1993a), and applying the infinity condition and the single-valuedness requirement, we may obtain a complete solution for \( f(z) \) (please refer to (Hwu and Fan, 1998) for detailed derivation) as

\[
f'(z) = -i \mathbf{A}^{-1} \mathbf{M}^{-1} \Lambda \left( \frac{z + 2iae_a}{z} \right) \chi_u(z) \Lambda^{-1} \mathbf{B} \mathbf{A}^T \mathbf{t}^x
\]

\[
(10a)
\]

where \( \mathbf{M} = -i \mathbf{B} \mathbf{A}^{-1} \) and

\[
\chi_u(z) = \frac{1}{\sqrt{z^2 - a^2}} \left( \frac{z - a}{z + a} \right)^{e_u}.
\]

\( e_u \) is the oscillatory index determined by the material elastic properties and \( \Lambda \) is an eigenvector matrix associated with this oscillatory index (Hwu, 1993a). The stress distribution along the interface
can also be obtained as
\[ t(x) = \mathbf{A}(x+2ia_{0}\mathbf{\chi}_{a}(x))\mathbf{A}^{-1}t^{\infty}, \quad |x| > a \]  
(11)

**A flat-ended punch:** (the punch is rigid)

Consider the indentation by a single flat-ended punch which makes contact with the half-plane over the region \(|x| \leq a\), and the force \(q\) applied on the punch is given. By using the general solutions obtained in the literature (Fan and Hwu, 1996), and applying the infinity condition and the force equilibrium condition of each punch, we may obtain a complete solution for \(f(z)\) (please refer to (Hwu and Fan, 1998) for detailed derivation) as
\[ f(z) = \frac{1}{2\pi} \mathbf{B}^{-1}\mathbf{\chi}_{a}(z)\mathbf{A}^{-1}q \]  
(12)

The contact pressure under the punch can also be obtained as
\[ t(x) = \frac{1}{\pi\sqrt{a^{2}-x^{2}}} [1 + \frac{1-c_{s}^{2}}{\beta^{2}} S^{2} + \frac{c_{s}}{\beta} S^{T} q], \quad |x| < a \]  
(13)

Since the stresses and deformations are real quantities, it is of interest to obtain the real form solutions in order to have a better understanding of the physical behavior of the punch problems. By using the identities obtained in the literature, the real form solution for the contact pressure under the punch and surface deformation gradient outside the punch can be found to be
\[ t(x) = \frac{1}{\pi\sqrt{a^{2}-x^{2}}} [1 + \frac{1-c_{s}^{2}}{\beta^{2}} S^{2} + \frac{c_{s}}{\beta} S^{T} q], \quad |x| < a \]  
(14a)

where \(\mathbf{L}\) and \(\mathbf{S}\) are 3x3 real matrices composed of the elasticity constants. They are defined by \(\mathbf{S} = i(2\mathbf{A}\mathbf{B}^{T} - 1)\) and \(\mathbf{L} = -2\mathbf{A}\mathbf{B}\mathbf{B}^{T}\). \(\beta\) is a real number defined as \(\beta = [-\frac{1}{2} \text{tr}(\mathbf{S}^{2})]^{1/2}\), and \(c_{s}, c_{j}, c_{s}', c_{j}'\) are real numbers defined as
\[ c_{s} = i\cosh(\pi\beta)e^{-\frac{a-x}{a}}, \quad c_{j} = e^{-\frac{a-x}{a}} \]  
(14b)

**Numerical example**

Consider an orthotropic half-plane whose material properties are \(E_{1} = 60.7\) Gpa, \(E_{2} = 24.8\) Gpa, \(G_{12} = 12\) Gpa, \(v_{12} = 0.23\), where \(E, G\) and \(v\) are, respectively, the Young's modulus, shear modulus and the Poisson's ratio. The subscript 1 and 2 denote the \(x\) and \(y\) directions. The contact region \(2a\) is set to be 2 m of which the size is just a reference for the infinite domain. The contact pressure and the surface deformation are shown in Figure 1. The stress singularity near the corners of the punch shown by eqn.(14) can be found in this figure. However, since the oscillatory zone is too small (about \(10^{-4}\)a), the oscillatory behavior near the corners of the punch can not be revealed by this figure. To see the stresses in depth and to see the anisotropic effect, the contour plot of the nondimensionalized stress shown in Figure 2 may be helpful, which shows that the maximum stress along \(x=\) constant occurs at a certain point under the surface not the point on the surface.

4. Concluding Remarks

In this report, the analogy between the interface crack problems and the punch problems is presented under the consideration that the materials above the interface and the punch are rigid. From the boundary conditions shown in (4) and (6) along the interface and the half-plane surface, we observe that the solutions to the interface crack problems and the punch problems should have the same forms with \(\mathbf{A}\) and \(\mathbf{B}\) interchanged. In addition to the analogy of the general solutions, the solution forms of the surface traction and the displacement gradient are also analogous to each other, which are shown in (8) and (9).

It should be emphasized that we state the analogy only by the solution form not the solution itself. Because for a real problem, the boundary conditions stated in (4) or (6) are not complete enough. For a finite body, there should be a condition describing the outer boundary. For an infinity body, the outer boundary condition is the so called infinity condition. Therefore, if we want to get an exactly analogous solution, all the boundary conditions should be analogous to each other. The examples presented in Section 3 show that the additional boundary conditions for a traction-free interface crack problem and for a flat-ended punch, are not exactly analogous to each other. Hence, solutions of \(f(z)\) found in (10) for a traction-free interface crack problem and in (12) for a flat-ended punch problem cannot be communicated by only interchanging \(\mathbf{A}\) and \(\mathbf{B}\).

Although (12) cannot be obtained by (10) with \(\mathbf{A}\) and \(\mathbf{B}\) interchanged, or vice versa, they really reveal the same oscillatory singularity behavior. From the stress distribution shown in (11) and (13), we observe that the oscillatory singularity characteristics is dominated by the function \(\chi_{a}(x)\) which is exactly the same for a
traction-free interface crack and a flat-ended punch.

The analogy given in this report is for a rigid punch and a rigid material above the interface. It is hoped that the analogy concept may be extended to the interface crack problems and the elastic contact problems between two dissimilar anisotropic media.

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References


計畫成果自評

本計畫成功地尋找到界面裂縫問題與壓陷問題之類比關係，這個關係在學術界是第一次提出，應會受到相當程度之重視。同時對從事實驗工作的學者亦應有所幫助，因若界面裂縫之實驗較難進行，可考慮進行與其類比之壓陷問題實驗。反之亦然。

所有計畫預期目標皆已達成，研究成果亦已送至國際知名期刊發表。