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ELLiptical Mixed Boundary Value Problems

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ABSTRACT

Recently, by combining the Stroh’s formalism with the method of analytical continuation, the mixed boundary valued problems with straight edges have been solved explicitly for general anisotropic media. These solutions are useful for engineering applications to composite materials, road pavements, geotechnical engineering and tribology, etc. To have a further understanding about the mixed boundary valued problems and to have a broad applications to engineering practices, we now extend this method to the problems with elliptical boundary edges, such as a pin-loaded hole in composite materials. Since no analytical solution can be found in the literature for general cases, the correctness of the derived solutions is verified by comparing the reduced special cases such as isotropic media. Representative numerical results are plotted for some cases to assist understanding of the physical behavior.

Keywords: Elliptical Mixed Boundary Value Problems, Anisotropic Elasticity, Stroh’s Formalism, Analytical Continuation Method
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CHAPTER 1
INTRODUCTION

Since the mechanics analysis of composite materials usually depends upon the model of linear anisotropic elasticity, the analytical study for the anisotropic elasticity becomes important for the understanding of the mechanical behavior of composite materials. Among various formulations, the complex variable formulations developed by Muskheishvili (1954) for isotropic elasticity, and by Lekhnitskii (1963, 1968) and Stroh (1958, 1962) for anisotropic elasticity, have been proved to be powerful mathematical tools dealing with the infinite domain problems. Through these methods, many exact closed form solutions have been obtained for the conventional elasticity problems, which are important for the engineers to catch the special feature of the discussed problems.

However, due to the mathematical infeasibility, most of the effort was put on the first and second fundamental boundary valued problems. For mixed boundary valued problems of anisotropic elasticity, very few works have been done. In my recent project (Hwu and Fan, 1994), a general procedure for solving the mixed boundary valued problems with straight edges has been established. With that general procedure, several examples have been solved completely, such as, a set of rigid punches of arbitrary profiles indenting into the surface of an anisotropic elastic half-plane with no slip occurring, and a sliding punch with or without friction.

In this project, we extend the Stroh formalism and the method of analytical continuation to the mixed boundary valued problems with elliptical boundary edges. In engineering practice, this kind of solutions may be useful for the understanding of the mechanical behavior of a pin-loaded hole in composite materials. By surveying the literature, it was found that this kind of problems were usually solved numerically by treating it as a first
fundamental problem with the pin-loaded pressure assumed to be a known function. The assumption of the pin-loaded pressure distribution may now be checked by using the analytical solution found in this project.

In earlier works, Muskheilishvili (1954) and England (1971) have developed a theoretical procedure by using the method of analytical continuation to deal with the mixed boundary value problem for an infinite isotropic body containing an elliptical hole which is loaded by a rigid attachment stamp on the hole boundary. Besides the analytical works stated in the above, most of the effort is devoted to the numerical calculation by finite element or boundary element methods. For instance, Chang (1986) utilized a special finite element to evaluate the effect of the assumed pin load distribution on the calculated strength and to predict failure mode of pin-loaded holes in laminated composites. Mahajerin and Sikarskie (1986) developed an efficient boundary element to study a loaded holes in composites. Wang (1994) investigate the effects of the clearance between pin and hole, the pin load magnitude, the contact condition and the relative pin/lug rigidities by using the finite element method.

It should be noted that the Stroh's formalism is developed under the assumption that all the material eigenvalues are distinct or all the independent material eigenvectors can be found when the eigenvalues are repeated. For degenerate materials such as isotropic materials whose material eigenvalues are repeated such that we can't find all the independent eigenvectors, the results should be modified in the analytical sense (Ting and Hwu, 1988). In numerical calculation, a small perturbation of the material eigenvalues are usually introduced to avoid additional work of formulation.
CHAPTER 2
TWO-DIMENSIONAL ANISOTROPIC ELASTICITY

2.1 Stroh's Formalism

Basic Equations

The basic equations for linear anisotropic elasticity are the strain-displacement equations, the stress-strain laws and the equations of equilibrium, which can be expressed in a fixed rectangular coordinate system $x_i$, $i = 1, 2, 3$ as (the symbols $x_1$ and $x_2$ will be replaced by $x$ and $y$ for two-dimensional problems in this dissertation)

$$
\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),
$$

$$
\sigma_{ij} = C_{ijkl} \varepsilon_{kl},
$$

$$
\sigma_{ij,j} = C_{ijkl} u_{k,sj} = 0,
$$

where $u_i$, $\sigma_{ij}$ and $\varepsilon_{ij}$ are respectively the displacement, stress and strain; the repeated indices imply summation; a comma stands for differentiation and $C_{ijkl}$ are the elastic constants which are assumed to be fully symmetric and positive definite.

In this report, we shall be concerned with two-dimensional elastic problems in which $x_3$ does not appear in the basic equations or the boundary conditions. These problems are of two distinct types. One is generalized plane strain problems, the other is generalized plane stress problems. The generalized plane strain problems are usually treated by employing $C_{ij}$ with the third row and third column deleted, or employing $\tilde{S}_{ij}(= C_{ij}^{-1})$ where

$$
\tilde{S}_{ij} = S_{ij} - S_{i3} S_{j3}/S_{33}.
$$

While the generalized plane stress problems are treated by employing $S_{ij}$ with the third row and third column deleted, or employing $\tilde{C}_{ij}(= S_{ij}^{-1})$ where

$$
\tilde{C}_{ij} = C_{ij} - C_{ij} C_{j3}/C_{33}.
$$
Note that the $6 \times 6$ matrices $C_{ij}$ and $S_{ij}$ are, respectively, the contracted notations of the elastic tensor $C_{ijkl}$ and $S_{ijkl}$ (Jones, 1975).

It is shown that the general solution to the above basic equations (2.1) may be expressed in terms of holomorphic functions of complex variables. This enables us to apply many of the powerful results of complex function theory to the two-dimensional elasticity.

For two-dimensional anisotropic elasticity, there are two different complex variable formulations in the literature. One is the Lekhnitskii approach (Lekhnitskii, 1963, 1968) which starts with the equilibrated stress functions followed by compatibility equations; the other is the Stroh's formalism (Stroh, 1958, 1962) which starts with the compatible displacements followed by equilibrium equations. Since both of them deal with the same basic equations given in (2.1), they should be equivalent. The equivalence and difference of these two formulations were discussed by Suo (1990) and Hwu (1996). Due to the neat and elegant feature, the Stroh's formalism will be employed in this report. For the convenience of readers' reference, we now briefly state the Stroh's formalism.

**General Solutions**

Consider a two-dimensional deformation in which $u_k$ $(k = 1, 2, 3)$ depends on $x$ and $y$ only, the general solution of $u_k$ to (2.1)\_3 has the form

$$u = a f(z), \quad z = x + py,$$

where $u = (u_1, u_2, u_3)$ is the vector form of displacement; $f$ is an arbitrary holomorphic function of its argument; $p$ and $a$ are, respectively, the eigenvalues and eigenvectors of the following eigenrelation

$$\left\{ \tilde{Q} + p(\tilde{R} + \tilde{R}^T) + p^2 \tilde{T} \right\} a = 0,$$

in which the superscript $T$ denotes the transpose, and the $3 \times 3$ matrices $\tilde{Q}$, $\tilde{R}$, $\tilde{T}$ are

$$Q_{ik} = C_{i1k1}, \quad R_{ik} = C_{i1k2}, \quad T_{ik} = C_{i2k2}.$$
There are six eigenvalues and eigenvectors from (2.3a). Since \( p \) cannot be real if the strain energy is positive (Eshelby et al., 1953), we have three pairs of complex conjugates for \( p \).

Let

\[
p_{\alpha+3} = \overline{p}_\alpha, \quad \text{Im} (p_\alpha) > 0, \quad \alpha = 1, 2, 3,
\]

where the overbar denotes the complex conjugate and Im stands for the imaginary part.

We then have

\[
g_{\alpha+3} = \overline{g}_\alpha, \quad \alpha = 1, 2, 3.
\]

Since the displacement \( \sim \) must be real, we let

\[
f_{\alpha+3} = \overline{f}_\alpha, \quad \alpha = 1, 2, 3.
\]

Thus the general solution for \( \sim \) can be written as

\[
\sim = 2 \Re \left\{ \sum_{\alpha=1}^{3} g_\alpha f_\alpha(z_\alpha) \right\}, \quad (2.4)
\]

in which \( \Re \) stands for the real part.

Introducing the vector

\[
b = (R^T + pT)\tilde{a} = -\frac{1}{p}(Q + pR)\tilde{a}, \quad (2.5)
\]

where the second equality comes from (2.3a), the stress \( \sigma_{ij} \) obtained by substituting (2.4) into (2.1)\(_{1,2}\) can be written as

\[
\sigma_{i1} = -\phi_{i,2}, \quad \sigma_{i2} = \phi_{i,1}, \quad (2.6)
\]

where \( \phi \) is the stress function

\[
\phi = 2 \Re \left\{ \sum_{\alpha=1}^{3} b_\alpha f_\alpha(z_\alpha) \right\}. \quad (2.7)
\]
More generally, if $\mathbf{t}$ is the surface traction vector, then

$$
\mathbf{t} = \frac{\partial \mathbf{\phi}}{\partial s},
$$

where $s$ is the arc length measured along a curved boundary in the direction such that when one faces the direction of increasing $s$, the material is located on the right hand side.

In this report, the method of analytical continuation (Muskeshilvili, 1954) plays an important role during the derivation of the analytical solutions. To fully utilize this method, we now rewrite the general solutions (2.4) and (2.7) into the following compact matrix form solution, i.e.

$$
\mathbf{u} = \mathcal{A} \mathbf{f}(z) + \mathcal{B} \bar{\mathbf{f}}(z),
$$

$$
\mathbf{\phi} = \mathcal{B} \mathbf{f}(z) + \mathcal{B} \bar{\mathbf{f}}(z),
$$

where

$$
\mathcal{A} = [a_1 \ a_2 \ a_3], \quad \mathcal{B} = [b_1 \ b_2 \ b_3],
$$

$$
\mathbf{f}(z) = [f_1(z_1) \ f_2(z_2) \ f_3(z_3)]^T.
$$

(The following paragraph is extracted from (Suo, 1990))

*Stated below is a trivial observation that makes analytic continuation arguments possible.*

A function $h(z)$ is an analytic function of $z = x + py$ for $y > 0$ (or $y < 0$) for any $p$ if it is analytic for $y > 0$ (or $y < 0$) for one $p$, where $p$ is any complex number with positive imaginary part.

Consequently, when talking about a function analytic in the upper (or lower) half plane, one needs not refer to its argument, as long as the argument has the form $z = x + py$ (Im $p > 0$).
Without loss of any information, we can and will present our solutions by the function vector \( \vec{f}(z) \) defined as

\[
\vec{f}(z) = [f_1(z), f_2(z), f_3(z)]^T, \tag{2.10}
\]

where the argument has the generic form \( z = x + py \) (\( \text{Im} \ p > 0 \)). Once the solution of \( \vec{f}(z) \) is obtained for a given boundary value problem, a replacement of \( z_1, z_2 \) or \( z_3 \) should be made for each component function to calculate field quantities from (2.9).

With the above statement, we know that the function vector \( \vec{f}(z) \) obtained through the method of analytical continuation has the form of (2.10) which is not consistent with the solution form shown in (2.9c) and is valid only along the boundary. To get the explicit full domain solution, a mathematical operation based upon the statement following (2.10) is needed. A translating technique presented by Hwu (1993) is then introduced as below.

If an implicit solution is written as

\[
\vec{f}(z) = \mathcal{C} \ll g_\alpha(z) \gg q, \tag{2.11}
\]

with the understanding the subscript of \( z \) is dropped before matrix product and a replacement of \( z_1, z_2 \) or \( z_3 \) should be made for each component function of \( \vec{f}(z) \) after the multiplication of matrices, the explicit solution can be expressed as

\[
\vec{f}(z) = \sum_{k=1}^{3} \ll g_k(z_\alpha) \gg \mathcal{C} \mathcal{I}_k q, \tag{2.12a}
\]

where

\[
\mathcal{I}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{I}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{I}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.12b}
\]

In the above, the angular brackets \( \ll \gg \) stands for the diagonal matrix in which each component is varied according to the Greek index \( \alpha \), i.e.

\[
\ll g_\alpha \gg = \text{diag} [g_1, g_2, g_3].
\]
Some Identities

The equations (2.3a) and (2.5) can be recast in the standard eigenrelation

\[ \widetilde{N} \xi = p \xi, \]  

(2.13)

where

\[ \widetilde{N} = \begin{bmatrix} \widetilde{N}_1 & \widetilde{N}_2 \\ \widetilde{N}_3 & \widetilde{N}_1^T \end{bmatrix}, \quad \xi = \begin{bmatrix} a \\ b \end{bmatrix}, \]

\[ \widetilde{N}_1 = -\widetilde{T}^{-1} \widetilde{R}^T, \]  

\[ \widetilde{N}_2 = \widetilde{T}^{-1}, \]  

\[ \widetilde{N}_3 = \widetilde{R} \widetilde{T}^{-1} \widetilde{R}^T - \widetilde{Q}. \]  

(2.14)

Since \( \widetilde{Q} \) and \( \widetilde{T} \) are symmetric, we see that \( \widetilde{N}_2 \) and \( \widetilde{N}_3 \) are also symmetric.

It has been proved that (Stroh, 1958; Ting, 1986) the material eigenvector matrices \( \widetilde{A} \) and \( \widetilde{B} \) have the following orthogonality relations

\[ \widetilde{A}^T \widetilde{B} + \widetilde{B}^T \widetilde{A} = \mathbb{I} = \widetilde{A}^T \widetilde{A} + \widetilde{B}^T \widetilde{B}, \]  

(2.15a)

\[ \widetilde{A}^T \widetilde{B} + \widetilde{B}^T \widetilde{A} = 0 = \widetilde{A}^T \widetilde{B} + \widetilde{B}^T \widetilde{A}. \]  

(2.15b)

\[ \widetilde{A} \widetilde{A}^T + \widetilde{A} \widetilde{A}^T = 0 = \widetilde{B} \widetilde{B}^T + \widetilde{B} \widetilde{B}^T, \]  

(2.15c)

\[ \widetilde{B} \widetilde{A}^T + \widetilde{B} \widetilde{A}^T = \mathbb{I} = \widetilde{A} \widetilde{B} + \widetilde{B} \widetilde{B}, \]  

(2.15d)

where \( \mathbb{I} \) is the unit matrix. Equations (2.15c) and (2.15d) imply that \( \widetilde{A} \widetilde{A}^T \), \( \widetilde{B} \widetilde{B}^T \) and \( (\widetilde{A} \widetilde{B}^T - \frac{1}{2} \mathbb{I}) \) are purely imaginary. Hence we let

\[ \mathcal{H} = 2i \widetilde{A} \widetilde{A}^T = \mathcal{H}^T, \]  

(2.16a)

\[ \mathcal{L} = -2i \widetilde{B} \widetilde{B}^T = \mathcal{L}^T, \]  

(2.16b)

\[ \mathcal{S} = i(2 \widetilde{A} \widetilde{B}^T - \mathbb{I}), \]  

(2.16c)
where \( H \), \( L \) and \( S \) are real, and it can be shown that \( H \), \( L \) are symmetric and positive definite if the strain energy is positive.

From (2.13) we see that for any integer \( n \), positive or negative, using the notation of (2.14) and (2.9(c))\(_{1,2} \), we have

\[
\mathcal{N}^n \begin{bmatrix} A \\ S \end{bmatrix} = \begin{bmatrix} A \ll p^n_\alpha \gg \\ S \ll p^n_\alpha \gg \end{bmatrix},
\]

(2.17)

If we postmultiply (2.17) by \( [\mathcal{A}^T, \mathcal{B}^T] \) and use (2.16), we obtain

\[
\mathcal{N}^n \begin{bmatrix} -iH & I - iS \\ I - iS^T & iL \end{bmatrix} = 2 \begin{bmatrix} A \ll p^n_\alpha \gg A^T \ll p^n_\alpha \gg \mathcal{A}^T \\ S \ll p^n_\alpha \gg A^T \ll p^n_\alpha \gg \mathcal{B}^T \\ B \ll p^n_\alpha \gg \mathcal{A}^T \\ B \ll p^n_\alpha \gg \mathcal{B}^T \end{bmatrix}.
\]

(2.18)

Carrying out the matrices product yields the following identities (Ting, 1988)

\[
2A \ll p^n_\alpha \gg A^T = N_2^{(n)} + i \left[ N_1^{(n)} H - N_2^{(n)} S^T \right],
\]
\[
2A \ll p^n_\alpha \gg B^T = N_1^{(n)} - i \left[ N_1^{(n)} S - N_2^{(n)} L \right] = N_1^{(n)} - i \left[ S N_1^{(n)} + H N_3^{(n)} \right],
\]
\[
2B \ll p^n_\alpha \gg B^T = N_3^{(n)} - i \left[ N_3^{(n)} S - N_1^{(n)} L \right],
\]

(2.19)

where \( N_i^{(n)} \) are the components of the matrix \( \mathcal{N}^n \).

**Degenerate Materials**

It should be noted that the general solutions shown in (2.9) are valid only for the nondegenerate materials, that is the material eigenvalues \( p_\alpha, \alpha = 1, 2, 3 \), are distinct or three independent material eigenvectors \( a_\alpha, b_\alpha, \alpha = 1, 2, 3 \), can be found when \( p_\alpha \) are repeated. For the degenerate materials such as isotropic materials, the complex eigenvector matrices \( \mathcal{A} \) and \( \mathcal{B} \) can not be obtained explicitly. Thus, the general solutions shown in (2.9) should be modified (Ting, 1982). However, if the final solutions do not contain any material eigenvalues \( p_\alpha \) or eigenvector matrices \( \mathcal{A}, \mathcal{B} \) explicitly, or they may be expressed in terms of the real fundamental elasticity matrices such as \( S, H, L \) and \( \mathcal{N}_i \) (Ting, 1988), they will be applicable to any kinds of anisotropic materials including the degenerate materials.
In numerical calculation for isotropic materials, a small perturbation of the material eigenvalues $\rho$ will be made, and then the corresponding eigenvector matrices $\widetilde{A}$ and $\widetilde{B}$ can be approximately obtained by (2.3a) and (2.5) respectively.

2.2 Boundary Conditions

The general solutions derived in the last section, (2.9), have satisfied all the basic equations listed in (2.1). The only unknown of (2.9) is the complex function vector $\hat{f}(z)$ which should be determined by the satisfaction of boundary conditions set for the physical problems. Of several physically distinct types of boundary conditions, there are three fundamental types of boundary conditions which seem to be of considerable physical interest. In the first it is supposed the surface traction are specified at all points along the boundary. Thus if $\widehat{t}$ is the prescribed traction value along the boundary $C$, the boundary conditions may be written as

$$\hat{t}(s) = \widehat{t}(s), \quad s \in C.$$  \hspace{1cm} (2.20a)

Or, by integration of (2.8),

$$\hat{\phi}(s) = \widehat{\phi}(s), \quad s \in C.$$ \hspace{1cm} (2.20b)

The problem to obtain a solution of the basic equations (2.1) subject to the boundary conditions (2.20) is referred to the stress boundary value problem.

Alternatively the displacement $\hat{u}$ may be specified at all points along the boundary, so that

$$\hat{u}(s) = \widehat{u}(s), \quad s \in C,$$ \hspace{1cm} (2.21)

which is referred to the displacement boundary value problem.

In many physical problems, all or a part of the stress boundary conditions hold over a part $C^*$ of $C$ and the displacements are defined over the remainder $C - C^*$ of $C$. That
is,

\[ \xi(s) = \hat{\xi}(s), \quad s \in C^*, \]
\[ \eta(s) = \hat{\eta}(s), \quad s \in C - C^*, \]

(2.22a)

or

\[ t_i(s) = \hat{t}_i(s), \quad i = 1, \text{ and/or } 2, \text{ and/or } 3, \]
\[ u_j(s) = \hat{u}_j(s), \quad j \neq i, \]

(2.22b)

which constitutes the mixed boundary value problems. In the literature, there are many discussions and solutions to the stress and displacement boundary value problems. Relatively few solutions are devoted to the mixed boundary value problems due to its rather more awkward nature than the other two. In this report, we like to study the mixed boundary value problems of which the boundary \( C \) is elliptical.

No matter what kinds of boundaries are considered, it is always necessary to examine the behavior of the stresses and displacements at infinity for the problems with infinite domain. Since this is an important problem for the later derivation, we discuss it in this section.

In general, the complex function vector \( \vec{f}(z) \) may be expressed as

\[ \vec{f}(z) = \sum_{k=1}^{n} \text{log} \left( z_{\alpha} - \tilde{z}_{\alpha}^{(k)} \right) \otimes \gamma_k + \vec{f}_0(z), \]

(2.23)

where \( \tilde{z}_{\alpha}^{(k)} \), \( k = 1, 2, \cdots, n \) are the points in the interior of the internal boundaries \( C_k \) (Figure 1); \( \gamma_k \) has a direct physical interpretation in terms of the resultant forces acting over the contours \( C_k \); \( \vec{f}_0(z) \) is a holomorphic function in the field. If we consider that the stresses are bounded at infinity, it may be shown that for large \( |z| \) (2.23) has the form

\[ \vec{f}(z) = \text{log} z_\alpha \otimes \gamma + \text{log} z_\alpha \otimes \eta + O(1), \]

(2.24)

where \( \eta \) is related to the stresses at infinity and can be found by considering the entire bodies under uniform stresses applied at infinity, whose solution is (Ting, 1988)

\[ \eta = A^T \xi^\infty_2 + B^T \xi^\infty_1, \]

(2.25)
where
\[
\mathbf{\xi}_2^\infty = \begin{Bmatrix} \sigma_{12}^\infty \\ \sigma_{22}^\infty \\ \sigma_{32}^\infty \end{Bmatrix}, \quad \mathbf{\xi}_1^\infty = \begin{Bmatrix} \varepsilon_{11}^\infty \\ \varepsilon_{12}^\infty \\ 2\varepsilon_{13}^\infty \end{Bmatrix}.
\]

Furthermore, the relation between \( \mathbf{\xi}_2^\infty \) and \( \mathbf{\xi}_1^\infty = (\sigma_{11}^\infty, \sigma_{21}^\infty, \sigma_{11}^\infty)^T \) is found as
\[
N_1^T \mathbf{\xi}_2^\infty + N_3 \mathbf{\xi}_1^\infty = -\mathbf{\xi}_1^\infty. \tag{2.26}
\]
\( \mathbf{\gamma} \) is related to the resultant forces applied to the outer boundary \( C_0 \) which may be determined as follows.

If \( \mathbf{\gamma} \) is the resultant applied force over all internal boundaries \( C_k \), then from (2.8) and (2.9b)
\[
\mathbf{\gamma} = \int_{C_0} \mathbf{\gamma} \ ds = \left[ Bf(z) + \overline{Bf(z)} \right]_{C_0}. \tag{2.27}
\]
Substituting (2.24) into (2.27), we have
\[
B \mathbf{\gamma} - \overline{B \mathbf{\gamma}} = \frac{\mathbf{\gamma}}{2\pi i}. \tag{2.28}
\]
With (2.24) and (2.9a), the requirement of single-valued displacement around the outer boundary \( C_0 \) leads to
\[
A \mathbf{\gamma} - \overline{A \mathbf{\gamma}} = 0. \tag{2.29}
\]
Through the use of the orthogonality relation shown in (2.15), the solution to (2.28) and (2.29) is found to be
\[
\mathbf{\gamma} = \frac{1}{2\pi i} A^T \mathbf{\gamma}. \tag{2.30}
\]

2.3 Hilbert Problem

To determine the unknown complex function vector \( \mathbf{f}(z) \) through the satisfaction of the boundary conditions stated in the above section, the method of analytical continuation (Muskheilishvili, 1954) is usually applied. By this method, the boundary conditions are
usually transformed to a mathematical problem called Hilbert problem of vector form (Muskeshlishvili, 1946) which is to determine a sectionally holomorphic function \( \tilde{\psi}(z) \), defined in the whole plane cut along \( L \) which is the union of a finite set of arcs \( L_k = (a_k, b_k) \), \( k = 1, 2, \ldots, n \), and satisfying the boundary condition

\[
\tilde{\psi}(t^+) - G\tilde{\psi}(t^-) = g(t), \quad t \in L, \tag{2.31}
\]

where \( G \) is a constant matrix. This problem is similar to the Hilbert problem of scalar form discussed in (Muskeshlishvili, 1954), which is

\[
\psi(t^+) - G\psi(t^-) = g(t), \quad t \in L, \tag{2.32}
\]

The main difference between (2.31) and (2.32) is that if \( G \) is not a diagonal matrix, (2.31) cannot be solved by only using the solution of (2.32) since each equation of (2.31) may look like

\[
\psi_i(t^+) - (G_{i1}\psi_1(t^-) + G_{i2}\psi_2(t^-) + G_{i3}\psi_3(t^-)) = g_i(t), \quad i = 1, 2, 3, \tag{2.33}
\]

which are coupled equations of \( \psi_1(t) \), \( \psi_2(t) \) and \( \psi_3(t) \). In Hwu’s paper (Hwu, 1992), a general solution for \( G = -\frac{1}{M}M^* \), where \( M^* \) is a bimaterial matrix, has been derived. Replacing \( -\frac{1}{M}M^* \) by \( G \) in Hwu’s solution, we may provide a general solution to (2.31) for any constant matrix \( G \), i.e.

\[
\tilde{\psi}(z) = \frac{1}{2\pi i} X_0(z) \int_L \frac{1}{t - z} [X_0^+(t)]^{-1} g(t) dt + X_0(z) P_n(z), \tag{2.34}
\]

where \( X_0(z) \) is the basic Plemelj function in which the branch can be set such that \( z^n X_0(z) \to I \) as \( |z| \to \infty \), and \( P_n(z) \) is a function to be determined by satisfying the conditions at infinity.
In this report, a special Hilbert problem will be derived with $G = -\overline{M}\overline{M}^{-1}$ where $M$ is the impedance matrix (Ingebrigtsen and Tonning, 1969)

\[
\overline{M} = -iB\overline{A}^{-1} = H^{-1}(I + i\overline{S}) = (I - i\overline{S}^T)H^{-1},
\]

\[
\overline{M}^{-1} = i\overline{A}\overline{B}^{-1} = L^{-1}(I + i\overline{S}^T) = (I - i\overline{S})L^{-1}.
\] (2.35)

The third equalities in (2.35) come from the fact that $H^{-1}\overline{S}$ and $\overline{S}L^{-1}$ are antisymmetric. Hence $\overline{M}$ and $\overline{M}^{-1}$ are Hermitian matrices (Ting, 1988). For the convenience of the following discussion, we now solve this special Hilbert problem completely.

For the present case, the basic Plemelj function $\overline{X}_0(z)$ in the solution of the Hilbert problem (2.34) satisfies

\[
\begin{align*}
\overline{X}_0^+(t) &= \overline{X}_0^-(t), \quad t \not\in L, \\
\overline{X}_0^+(t) + \overline{M}\overline{M}^{-1}\overline{X}_0^-(t) &= 0, \quad t \in L,
\end{align*}
\] (2.36)

i.e.

\[
\overline{X}_0(z) = \Lambda \Gamma(z),
\] (2.37a)

where

\[
\Lambda = [\lambda_1, \lambda_2, \lambda_3],
\]

\[
\Gamma(z) = \left< \prod_{j=1}^{n} (z - a_j)^{-1 + \delta_\alpha} (z - b_j)^{\delta_\alpha} \right>.
\] (2.37b)

$\delta_\alpha$ and $\lambda_\alpha$, $\alpha = 1, 2, 3$ of (2.37b) are the eigenvalues and eigenvectors of

\[
(\overline{M}^{-1} + e^{2\pi i \delta}\overline{M}^{-1}) \overline{\lambda} = \overline{0}.
\] (2.38)

To obtain the explicit form of the eigenvalues $\delta$, we can substitute (2.35) into (2.38), then for a nontrivial solution of $\overline{\lambda}$ we obtain

\[
\| (1 - e^{2\pi i \delta}) \overline{S}L^{-1} + i(1 + e^{2\pi i \delta})L^{-1} \| = 0.
\] (2.39)

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Since $L^{-1}$ is positive definite, the determinant is nonzero if we set $\delta = 0$. Hence $(1 - e^{2\pi i \delta}) \neq 0$, and (2.39) may be represented as

$$\| \tilde{S} L^{-1} + i \beta L^{-1} \| = 0,$$

(2.40)

where

$$\beta = \frac{1 + e^{2\pi i \delta}}{1 - e^{2\pi i \delta}}.$$

Rearranging the above relation leads to

$$e^{2\pi i \delta} = \frac{1 - \beta}{1 + \beta}.$$

It can be shown that if $\delta$ is a root, so is $\delta + n$ where $n$ is an integer. If only $-1 < \text{Re}(\delta) \leq 0$ are considered (Ting, 1986), we have

$$\delta = -\frac{1}{2} + \frac{i}{2\pi} \ln \frac{1 + \beta}{1 - \beta}.$$

(2.41)

The theorem proved by Ting (1986) states that:

Let $\beta$ be a root of the $3 \times 3$ determinant

$$\| \tilde{W} + i \beta \tilde{D} \| = 0,$$

where $\tilde{D}$ is a real, symmetric and positive definite matrix, while $\tilde{W}$ is a real antisymmetric matrix, then

$$-\frac{1}{2} \text{tr}(\tilde{W} \tilde{D}^{-1})^2 > 0,$$

where tr stands for the trace of matrix, and the three roots are all real given by

$$\beta = 0, \quad \beta = \pm \left[-\frac{1}{2} \text{tr}(\tilde{W} \tilde{D}^{-1})^2\right]^{\frac{1}{3}}.$$
By the above theorem and the fact that $\Sigma L^{-1}$ is real and antisymmetric and $L^{-1}$ is real, symmetric and positive definite, we can obtain the solution for the eigenproblem (2.38) by replacing $W, D$ by $\Sigma L^{-1}, L^{-1}$, i.e.

$$\delta_\alpha = -\frac{1}{2} + i\epsilon_\alpha, \quad \alpha = 1, 2, 3,$$

(2.42a)

where

$$\epsilon_1 = \epsilon = \frac{1}{2\pi} \ln \frac{1 + \beta}{1 - \beta}, \quad \epsilon_2 = -\epsilon, \quad \epsilon_3 = 0,$$

(2.42b)

$$\beta = [-\frac{1}{2} \text{tr}(\Sigma^2)]^{1/2}.$$

Moreover, for normalizing the eigenvector matrix $\Lambda$, the normalization proposed by Hwu (1993) may be slightly changed to fit the present case, i.e.

$$\frac{1}{2} \Lambda^T (\mathcal{M}^{-1} + \overline{\mathcal{M}}^{-1}) \Lambda = I.$$

(2.43)

**Evaluation of Line Integrals**

In the solution (2.34) of the Hilbert problem, the following line integral occurs

$$j(z) = \int_L \frac{1}{t - z} [X_0^+(t)]^{-1} g(t) dt.$$

(2.44)

Suppose that $g(t)$ is a polynomial, a situation which often occurs in practice. Then the integral along each segment $L_k$ may be expressed in terms of an integral along a lacet $C_k$ surrounding $L_k$ as shown in Figure 2, and assume that $z$ remains outside these lacets.

The contour integral around the lacet $C_k$ may be represented as

$$\int_{C_k} \frac{1}{\zeta - z} [X_0(\zeta)]^{-1} g(\zeta) d\zeta = \int_{L_k} \frac{1}{t - z} [X_0^+(t)]^{-1} g(t) dt - \int_{L_k} \frac{1}{t - z} [X_0^-(t)]^{-1} g(t) dt$$

$$+ \lim_{\rho \to 0} \int_{|\zeta - a_k| = \rho} \frac{1}{\zeta - z} [X_0(\zeta)]^{-1} g(\zeta) d\zeta$$

$$+ \lim_{\rho \to 0} \int_{|\zeta - b_k| = \rho} \frac{1}{\zeta - z} [X_0(\zeta)]^{-1} g(\zeta) d\zeta.$$

(2.45)
It can be shown that the last two integrals of (2.45) tend to zero as \( \rho \to 0 \). Hence from (2.36)_2

\[
\int_{C_k} \frac{1}{\zeta - z} [X_0(\xi)]^{-1} g(\xi) d\xi = \int_{L_k} \frac{1}{t - z} [X_0^+(t)]^{-1} (I + \overline{M}M^{-1}) g(t) dt ,
\]

then the integral \( \hat{j}(z) \) may be expressed in the form

\[
\hat{j}(z) = \int_C \frac{1}{\zeta - z} [X_0(\xi)]^{-1} (I + \overline{M}M^{-1})^{-1} g(\xi) d\xi ,
\]

where \( C \) is the union of the lacets \( C_1, C_2, \ldots, C_n \).

Replacing \( C \) by a counterclockwise circle contour \( C_\infty \) at a large distance \( R \), we can obtain that

\[
\hat{j}(z) = 2\pi i S - \int_{C_\infty} \frac{1}{\zeta - z} [X_0(\xi)]^{-1} (I + \overline{M}M^{-1})^{-1} g(\xi) d\xi ,
\]

where \( S \) is the sum of residues of the poles of the integrand in (2.47) lying between \( C \) and \( C_\infty \). The second term has the form

\[
\lim_{R \to \infty} \int_0^{2\pi} \frac{R e^{i\theta}}{R e^{i\theta} - z} [X_0(Re^{i\theta})]^{-1} (I + \overline{M}M^{-1})^{-1} g(Re^{i\theta}) i d\theta ,
\]

where \( R \) is the radius of the contour \( C_\infty \). It can be shown that only terms independent of \( Re^{i\theta} \) can contribute to the above integral. Then with a given function \( g(t) \), the integral \( \hat{j}(z) \) can be explicitly evaluated from (2.48) and (2.49).

For example, consider an integral along a single line \( L = (-a, a) \) and let \( g(t) = \xi \), where \( \xi \) is a given constant vector. The sum of the residues is

\[
S = [X_0(z)]^{-1} (I + \overline{M}M^{-1})^{-1} \xi .
\]

To calculate the integral shown in (2.49), by (2.37b) we can express \( \Gamma^{-1}_\alpha(\xi) \) for large \( |\xi| \) as

\[
\Gamma^{-1}_\alpha(\xi) = (\xi + a)^{1+\delta_\alpha}(\xi - a)^{-\delta_\alpha}
\]

\[
= \xi + 2i a \epsilon_{\alpha} + O\left(\frac{1}{\xi}\right) , \quad \alpha = 1, 2, 3 .
\]
Then (2.49) becomes

\[
\lim_{R \to \infty} \int_{0}^{2\pi} (1 + \frac{z}{R e^{i\theta}} + \cdots) \ll Re^{i\theta} + 2ia\epsilon + \cdots \gg i \, d\theta \sim \Lambda^{-1}(\sim + \bar{M}M^{-1})^{-1} \sim \\
= 2\pi i \ll z + 2ia\epsilon \gg \Lambda^{-1}(\sim + \bar{M}M^{-1})^{-1} \sim 
\]

(2.51)

From (2.48), (2.50) and (2.51) we obtain the final result of \( \tilde{j}(z) \) as

\[
\tilde{j}(z) = 2\pi i \left\{ [X_0(z)]^{-1} - \ll z + 2ia\epsilon \gg \Lambda^{-1} \right\} (\sim + \bar{M}M^{-1})^{-1} \sim 
\]

(2.52)

Similarly, if \( g(t) = tg \), the integral \( \tilde{j}(z) \) can be evaluated as

\[
\tilde{j}(z) = 2\pi i \left\{ z[X_0(z)]^{-1} - \ll z^2 + 2ia\epsilon z - \left( \frac{1}{2} + 2\epsilon^2 \right) a^2 \gg \Lambda^{-1} \right\} (\sim + \bar{M}M^{-1})^{-1} \sim 
\]

(2.53)
CHAPTER 3

GENERAL ANALYTICAL SOLUTIONS

In the theory of two-dimensional linear elasticity, analytical solutions of mixed boundary value problems for awkwardly shaped regions are difficult to be directly obtained. Therefore, the conformal mapping technique is employed to transform the region into one of simpler shape in which the solution is easily found. In general, the boundary conditions become more cumbersome in the transformed region, however, the difficulties encountered to handle the boundary conditions are easier than those in treating the problem in the original region.

In this chapter, we consider the mixed boundary value problems in an infinite region with an interior elliptical boundary which will be mapped onto a unit circle by a conformal mapping function. Then the method of analytical continuation is employed to deal with the boundary conditions in the transformed domain. A solution can be obtained through the Hilbert problem which is formulated by applying the method of analytical continuation to the boundary conditions. For illustration, some special cases are studied in detail to verify the formulation derived in this chapter.

3.1 Conformal Mapping

Consider a transformation

\[ z_\alpha = m_\alpha(\zeta_\alpha), \]  

which maps the points along an elliptical boundary of an infinite anisotropic elastic body in the \( z_\alpha \)-domain onto a unit circle in the \( \zeta_\alpha \)-domain. To ensure this mapping between the \( z_\alpha \)-domain and \( \zeta_\alpha \)-domain is one-to-one and invertible, the transformation is assumed to be single-valued by selecting a particular branch if needed. In addition, to preserve the
complex variable formulation, \( m_\alpha(\zeta_\alpha) \) must be holomorphic in the \( \zeta_\alpha \)-domain and that \( m_\alpha'(\zeta_\alpha) \neq 0 \) for all points in the mapped domain.

Now suppose that the geometry of the elliptical boundary considered can be represented as

\[
x = a \cos \psi, \quad y = b \sin \psi, \tag{3.2}
\]

where \( a, b \) are the lengths of semi-axes of the elliptical hole. By setting the lengths of the major and minor axes of the elliptical hole to be equal, a circular hole can be obtained. By approaching the length of the minor axis to zero, a crack can be considered to be a special case of the elliptical hole.

To map the elliptical boundary represented by (3.2) onto a unit circle \( |\zeta| = 1 \), the mapping function can be found as (Hwu, 1990)

\[
z_\alpha = \frac{1}{2} \left\{ (a - ibp_\alpha)\zeta_\alpha + (a + ibp_\alpha) \frac{1}{\zeta_\alpha} \right\}, \tag{3.3a}
\]

or

\[
\zeta_\alpha = \frac{z_\alpha + \sqrt{z_\alpha^2 - (a^2 + b^2 p_\alpha^2)}}{a - ib p_\alpha}, \tag{3.3b}
\]

in which the branch cut is selected so that \( |\zeta_\alpha| \to \infty \) as \( |z_\alpha| \to \infty \). To check whether the mapping function (3.3) is conformal, we must find that the locations of the roots \( \zeta_\alpha^\circ \) of the equation \( m_\alpha'(\zeta_\alpha) = 0 \), i.e.

\[
(a - ip_\alpha b) - (a + ip_\alpha b)\zeta_\alpha^{\circ -2} = 0, \tag{3.4a}
\]

or

\[
\zeta_\alpha^\circ = \pm \sqrt[4]{\frac{a + ibp_\alpha}{a - ibp_\alpha}}. \tag{3.4b}
\]

If we consider an infinite region containing an interior elliptical opening, the requirement for a conformal mapping is that all the roots of (3.4) are located inside the unit circle. If
$p_{\alpha I}, p_{\alpha R}$ are, respectively, the real and imaginary parts of $p_{\alpha}$, the absolute value of $\zeta_{\alpha}^2$ is
\[
|\zeta_{\alpha}^2| = \frac{(a - b p_{\alpha I})^2 + (b p_{\alpha R})^2}{(a + b p_{\alpha I})^2 + (b p_{\alpha R})^2}.
\]
Since $p_{\alpha I} > 0$ and $0 < b \leq a$, we have $|\zeta_{\alpha}^2| < 1$ which leads to $|\zeta_{\alpha}| < 1$. The roots are therefore located inside the unit circle $|\zeta_{\alpha}| = 1$ and the transformation function (3.3) is single-valued and conformal outside the elliptic hole.

3.2 Analytical Solutions for General Cases

By using the mapping function discussed in the previous section, the mixed boundary value problems for an infinite anisotropic body containing an elliptical hole can now be solved in the mapped domain. By employing the method of analytical continuation, the boundary conditions stated in (2.22) can be converted into a Hilbert problem. A solution can then be obtained from such a problem.

Consider the following boundary conditions as a representative case
\[
\tilde{\tau}^{m}(\sigma) = 0, \quad \sigma \notin L, \quad \tilde{\nu}^{n}(\sigma) = \tilde{\nu}^{r}(\sigma), \quad \sigma \in L, \tag{3.5}
\]
where $\sigma = e^{i\psi}$ denotes the points located on the unit circle of the mapped domain; $L$ is the union of a finite set of segments $L_k = (a_k, b_k), \, k = 1, 2, \ldots, n$, of the unit circle; $\tilde{\tau}^{m}$ is the surface traction along the boundary of which the normal is $\tilde{\nu}$; $\tilde{\nu}^{r}(\sigma)$ is the prescribed function of the displacement gradient along the tangent direction $\tilde{\nu}$; $\tilde{\nu}^{T}(\theta) = (-\sin \theta, \cos \theta, 0), \quad \tilde{\nu}^{T}(\theta) = (\cos \theta, \sin \theta, 0)$ are respectively the unit vectors normal and tangent to the boundary, the angle $\theta$ is measured counterclockwise from the positive $x$-axis to the tangent vector $\tilde{\nu}$ (Figure 3).

We denote the region exterior to the unit circle in the $\zeta_{\alpha}$-domain by $S^{-}$ and interior region by $S^{+}$, and $\sigma^{\pm}$ are the values of $\zeta$ approaching the boundary from $S^{\pm}$. Then the
traction $\tilde{t}_m$ on the surface of the boundary can be represented by (2.8) and (2.9b) as

$$\tilde{t}_m = \frac{\partial \phi}{\partial n} = \frac{\partial}{\partial n} \left[ B f(\zeta) + \overline{B f(\zeta)} \right], \quad \zeta \to \sigma^{-}. \quad (3.6)$$

To perform the differentiation in (3.6) on the boundary points of the mapped unit circle, a chain rule is applied. By the fact that $\frac{\partial}{\partial \zeta} f(\zeta) = 0$ if $f(\zeta)$ is a holomorphic function, we have

$$\tilde{t}_m = \frac{\partial}{\partial \zeta} \left[ B f(\zeta) \right] \frac{\partial \zeta}{\partial \psi} \frac{\partial \psi}{\partial \sigma} \left[ \frac{\partial z_\alpha}{\partial x} \frac{\partial x}{\partial n} + \frac{\partial z_\alpha}{\partial y} \frac{\partial y}{\partial n} \right] + \frac{\partial}{\partial \zeta} \left[ \overline{B f(\zeta)} \right] \frac{\partial \overline{\zeta}}{\partial \psi} \frac{\partial \psi}{\partial \sigma} \left[ \frac{\partial \overline{z}_\alpha}{\partial x} \frac{\partial x}{\partial n} + \frac{\partial \overline{z}_\alpha}{\partial y} \frac{\partial y}{\partial n} \right], \quad (3.7a)$$

where

$$\frac{\partial \zeta}{\partial \psi} = i e^{i \psi} = i \sigma, \quad \frac{\partial x}{\partial n} = \cos \theta, \quad \frac{\partial y}{\partial n} = \sin \theta, \quad \frac{\partial z_\alpha}{\partial x} = 1, \quad \frac{\partial z_\alpha}{\partial y} = p_\alpha. \quad (3.7b)$$

In order to evaluate the term $\frac{\partial z_\alpha}{\partial \psi}$ in (3.7a), the relation between $\theta$ and $\psi$ is introduced, i.e.

$$\frac{\partial x}{\partial \psi} = -\rho \cos \theta, \quad \frac{\partial y}{\partial \psi} = -\rho \sin \theta, \quad (3.8a)$$

where $x$ and $y$ are related to $\psi$ by (3.2). From (3.8a), we have

$$\rho = \sqrt{\left( \frac{\partial x}{\partial \psi} \right)^2 + \left( \frac{\partial y}{\partial \psi} \right)^2}. \quad (3.8b)$$

Knowing that $z_\alpha = x + p_\alpha y$, and using (3.8a), we obtain

$$\frac{\partial z_\alpha}{\partial \psi} = -\rho (\cos \theta + p_\alpha \sin \theta). \quad (3.8c)$$

By (3.7) and (3.8), equation (3.6) can be rewritten as

$$\tilde{t}_m = \lim_{\zeta \to \sigma^{-}} \left\{ -\frac{i}{\rho} \zeta B f'(\zeta) + \frac{i}{\rho} \frac{1}{\overline{\zeta}} \overline{B f'(\zeta)} \right\}, \quad (3.9)$$

where $'$ denotes differentiation with respect to its argument. By the following relation

$$\lim_{\zeta \to \sigma^{-}} \frac{f'(\zeta)}{\zeta} = \lim_{\zeta \to \sigma^{+}} \frac{1}{\zeta} f'(\overline{\zeta}),$$

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(3.9) can be further expressed as

\[ t_m = -\frac{i}{\rho} B f' (\sigma^-) + \frac{i}{\rho - B f' (\frac{1}{\sigma^+})} \]

(3.11)

Based upon the concept of the method of analytical continuation, we may introduce \( \tilde{\theta}' (\zeta) \) such that

\[ \tilde{\theta}' (\zeta) = \begin{cases} \zeta B f' (\zeta), & \zeta \in S^- \cr \frac{1}{\zeta B} f' (\frac{1}{\zeta}), & \zeta \in S^+ \end{cases} \]

(3.11)

Since \( f' (\zeta) \) should be holomorphic in the region \( S^- \), and by the theory of complex variable functions, \( f' (\frac{1}{\zeta}) \) would also be holomorphic in \( S^+ \). Hence \( \tilde{\theta}' (\zeta) \) introduced in (3.11) is sectionally holomorphic in the whole plane.

By the definition given in (3.11) and the following notation,

\[ \lim_{\zeta \to \sigma^-} \tilde{\theta}' (\zeta) = \tilde{\theta}' (\sigma^-), \quad \lim_{\zeta \to \sigma^+} \tilde{\theta}' (\zeta) = \tilde{\theta}' (\sigma^+) \]

(3.10) can be expressed as

\[ \tilde{\theta}' (\sigma^+) - \tilde{\theta}' (\sigma^-) = -i \rho \frac{1}{\mu} \mu_n, \quad \sigma \notin L \]

(3.12)

In a similar way, the boundary values of the displacement gradients along the tangent direction \( n \) of the segments \( L \) can be formulated as

\[ \tilde{\theta}' (\sigma^+) + M M^{-1} \tilde{\theta}' (\sigma^-) = -\rho \frac{1}{\mu} \mu_n, \quad \sigma \in L \]

(3.13)

Substituting (3.12) and (3.13) into the boundary conditions (3.5), we obtain the following Hilbert problem

\[ \tilde{\theta}' (\sigma^+) - \tilde{\theta}' (\sigma^-) = 0, \quad \sigma \notin L \]

\[ \tilde{\theta}' (\sigma^+) + M M^{-1} \tilde{\theta}' (\sigma^-) = -\rho \frac{1}{\mu} \mu_n, \quad \sigma \in L \]

(3.14)
By (2.34), the solution to this Hilbert problem is

\[
\tilde{\theta}'(\zeta) = -\frac{1}{2\pi i} X_0(\zeta) \int \frac{\rho}{t-\zeta} [X_0^+(t)]^{-1} \bar{M} \bar{a}'(t) dt + X_0(\zeta) \hat{p}(\zeta),
\]  

(3.15)

where \(X_0(\zeta)\) is the basic Plemelj function given in (2.36), and \(\hat{p}(\zeta)\) will be determined by the loading conditions.

Moreover, by (3.3), (3.11) and (2.24), the conditions of \(\tilde{\theta}'(\zeta)\) at infinity and zero point can be respectively represented as

\[
\tilde{\theta}'(\zeta) = B \bar{g} k a_3 \zeta^k + B \bar{g} a_1 \zeta + B \bar{\gamma} + O(\zeta^{-1}), \quad \text{as } |\zeta| \to \infty,
\]  

(3.16)

and

\[
\tilde{\theta}'(\zeta) = \bar{B} \bar{g} k \frac{\bar{a}_3}{\zeta^k} + \bar{B} \bar{g} \frac{\bar{a}_1}{\zeta} + \bar{B} \bar{\gamma} + O(\zeta), \quad \text{as } |\zeta| \to 0,
\]  

(3.17)

where \(\bar{g}\) and \(\bar{\gamma}\) are related to the loading conditions by (2.25) and (2.30).

Now the problem is solved in principle. In the next chapter, we will investigate special cases for illustration.
CHAPTER 4
RIGID STAMP INDENTATION

4.1 General Loading Conditions

Consider an infinite anisotropic body containing an elliptical hole of which the boundary can be expressed as shown in (3.2). From (3.8a,b), we have

\[ \rho \cos \theta = a \sin \psi, \quad \rho \sin \theta = -b \cos \psi, \]
\[ \rho = \sqrt{a^2 \sin^2 \psi + b^2 \cos^2 \psi}. \]  \hspace{1cm} (4.1)

Suppose that the hole is loaded by a rigid stamp along the segment which is mapped onto an arc \( L = (e^{-i\phi}, e^{i\phi}) \) in the \( \zeta_\alpha \)-domain. The profile of the stamp is assumed to be the same as the profile of the hole (Figure 4). Then the boundary conditions for the problem stated above may be written as

\[ \hat{t}_m(\sigma) = 0, \quad \sigma \not\in L, \]
\[ \hat{u}_n(\sigma) = 0, \quad \sigma \in L. \]  \hspace{1cm} (4.2)

Employing (3.15) with \( \hat{\zeta}' = 0 \), the solution to the present problem can be obtained as

\[ \hat{\theta}'(\zeta) = \Lambda \Gamma(\zeta) \hat{\rho}(\zeta), \]  \hspace{1cm} (4.3a)

where

\[ \Gamma(\zeta) = \left( \zeta - e^{-i\phi} \right)^{\frac{1}{2} + i\alpha} \left( \zeta - e^{i\phi} \right)^{-\frac{1}{2} + i\alpha}, \]

and the branch of \( \Gamma(\zeta) \) is selected so that \( \zeta \Gamma(\zeta) \to I \) as \( |\zeta| \to \infty \). By comparing (4.3) with the conditions of \( \hat{\theta}'(\zeta) \) at infinity and zero point, i.e. (3.16) and (3.17), it is apparent that \( \hat{\rho}(\zeta) \) can only have the form

\[ \hat{\rho}(\zeta) = d_2 \zeta^2 + d_1 \zeta + d_0 + d_{-1} \zeta^{-1}. \]  \hspace{1cm} (4.4)
Moreover, \( \tilde{\Gamma}(\zeta) \) can be expanded for large \(|\zeta|\) as

\[
\tilde{\Gamma}(\zeta) = \ll \frac{1}{\zeta} + \left[ \left( \frac{1}{2} - i\epsilon_\alpha \right)e^{i\phi} + \left( \frac{1}{2} + i\epsilon_\alpha \right)e^{-i\phi} \right] \frac{1}{\zeta^2} + O\left( \frac{1}{\zeta^3} \right) \gg ,
\]  
(4.5a)

and for small \(|\zeta|\) as

\[
\tilde{\Gamma}(\zeta) = \ll -e^{-2\phi_\alpha} \left\{ 1 + \left[ \left( \frac{1}{2} + i\epsilon_\alpha \right)e^{i\phi} + \left( \frac{1}{2} - i\epsilon_\alpha \right)e^{-i\phi} \right] \zeta + O(\zeta^2) \right\} \gg .
\]  
(4.5b)

Substituting (4.4) and (4.5) into (4.3), and using (3.16) and (3.17) for comparison, we obtain

\[
a_1 \tilde{B} \tilde{q} = \Lambda \tilde{d}_2 ,
\]

\[
\tilde{B} \tilde{\gamma} = \Lambda \tilde{d}_1 + \Lambda \ll \left( \frac{1}{2} - i\epsilon_\alpha \right)e^{i\phi} + \left( \frac{1}{2} + i\epsilon_\alpha \right)e^{-i\phi} \gg \tilde{d}_2 ,
\]  
(4.6a)

and

\[
\tilde{a}_1 \tilde{B} \tilde{q} = -\Lambda \ll e^{-2\phi_\alpha} \gg \tilde{d}_{-1} ,
\]

\[
\tilde{B} \tilde{\gamma} = -\Lambda \left\{ \ll e^{-2\phi_\alpha} \gg \tilde{d}_0 + \ll e^{-2\phi_\alpha} \left[ \left( \frac{1}{2} + i\epsilon_\alpha \right)e^{i\phi} + \left( \frac{1}{2} - i\epsilon_\alpha \right)e^{-i\phi} \right] \gg \tilde{d}_{-1} \right\} .
\]  
(4.6b)

Once the loading conditions are given, i.e., \( \tilde{q} \) and \( \tilde{\gamma} \) are given, the four unknowns \( \tilde{d}_{-1} \), \( \tilde{d}_0 \), \( \tilde{d}_1 \) and \( \tilde{d}_2 \) can be obtained by solving (4.6a) and (4.6b). Thus the problem is solved.

### 4.2 A Special Loading Condition

Now consider a particular loading condition that all the stresses vanish at infinity and a resultant force \( \tilde{P} = (X, Y, 0)^T \) is applied on the rigid stamp. In this case, the four unknowns \( \tilde{d}_i \), \( i = -1, 0, 1, 2 \), can be determined by using \( \tilde{q} = \tilde{0} \) and \( \tilde{\gamma} = \frac{1}{2\pi i} \tilde{A}^T \tilde{P} \). The results are

\[
\tilde{d}_{-1} = \tilde{0} ,
\]

\[
\tilde{d}_0 = -\ll e^{2\phi_\alpha} \gg \Lambda^{-1} \tilde{B} \tilde{\gamma} ,
\]

\[
\tilde{d}_1 = \Lambda^{-1} \tilde{B} \tilde{\gamma} ,
\]

\[
\tilde{d}_2 = \tilde{0} .
\]  
(4.7)
Hence the solution of $\tilde{J}'(\zeta)$ can be found by combining (3.11), (4.3), (4.4) and (4.7), which is

$$\tilde{J}'(\zeta) = B^{-1} \Lambda \Gamma(\zeta) \left[ \Lambda^{-1} B \gamma + \frac{-e^{2\phi_\alpha}}{\zeta} \right].$$

(4.8)

4.3 Isotropic Media

Since the only analytical solution related to the present problem is found for the cases of isotropic media, we now try to reduce our solution to this special case by using the correspondence relation proposed by Hwu (1996).

First, we must convert (4.8) into the explicit full field solution by using the translating technique stated in (2.11) and (2.12). The result is

$$\tilde{J}'(\zeta) = \sum_{k=1}^{3} \left[ \left\langle \dot{\gamma}_{k}(\zeta) \right\rangle B^{-1} \Lambda I_{k} \Lambda^{-1} B \gamma + \left\langle \dot{\gamma}_{k}(\zeta) - \frac{e^{2\phi_\alpha}}{\zeta} \right\rangle B^{-1} \Lambda I_{k} \Lambda^{-1} B \gamma \right].$$

(4.9)

Using the result of (4.9) and applying the correspondence relation between isotropic and anisotropic elasticity proposed by Hwu (1996), the complex potential function $\varphi'(\zeta)$ (Muskelelivili, 1954) of isotropic media can be found to be

$$\varphi'(\zeta) = -\frac{X + iY}{2\pi(1 + \kappa)} \left( 1 + \frac{\kappa e^{-2\phi_\epsilon}}{\zeta} \right) (\zeta - e^{-i\phi})^{-\frac{1}{2} + i\epsilon}(\zeta - e^{i\phi})^{-\frac{1}{2} - i\epsilon},$$

(4.10)


4.4 Numerical examples

In the present numerical example, our attention will focus on the hoop stress along the hole boundary. By using the explicit full field solution given in (4.9) and the equation (2.9b), the hoop stress $\sigma_{nn}$ along the hole boundary can be calculated by

$$\sigma_{nn} = -n^T(\theta)\dot{\varphi}_{,n,m}.$$

(4.11)
During the numerical calculation, we encounter a problem about the determination of the multi-valued function $\Gamma(\zeta) = \ll (\zeta - e^{-i\phi})^{-\frac{1}{2}} - i\epsilon_\alpha (\zeta - e^{i\phi})^{-\frac{1}{2} + i\epsilon_\alpha} \gg$. To deal with this problem, a special branch cut system has to be established such that the multi-valued function $\Gamma(\zeta)$ keeps continuous across the whole boundary $C: |\zeta| = 1$ except the segment $L$ in which $\tilde{\Gamma}(\zeta)$ induces a jump. For this purpose, we construct a bi-polar coordinate system (see Figure 5a,b), and let

$$\zeta - e^{-i\phi} = r_1 e^{i\eta_1}, \quad \zeta - e^{i\phi} = r_2 e^{i\eta_2}.$$ 

$\tilde{\Gamma}(\zeta)$ can then be expressed as

$$\tilde{\Gamma}(\zeta) = \ll (r_1 r_2)^{\frac{1}{2}} \left( \frac{r_2}{r_1} \right)^{i\epsilon_\alpha} e^{-\frac{1}{2} (\eta_1 + \eta_2)} e^{i\epsilon_\alpha (\eta_1 - \eta_2)} \gg .$$ (4.12)

For a boundary point $\sigma = e^{i\psi}$, the branch cut for the term $(\zeta - e^{-i\phi})^{-\frac{1}{2} - i\epsilon_\alpha}$ will be set such that (see Figure 5a)

$$\arg(\sigma^- - e^{-i\phi}) = \eta_1^*, \quad \arg(\sigma^+ - e^{-i\phi}) = \eta_1^* - 2\pi,$$ (4.13)

across the whole boundary $C$. On the other hand, the branch cut for $(\zeta - e^{i\phi})^{-\frac{1}{2} + i\epsilon_\alpha}$ will be set such that (see Figure 5b)

$$\arg(\sigma^- - e^{i\phi}) = \eta_2^*, \quad \arg(\sigma^+ - e^{i\phi}) = \eta_2^* - 2\pi, \quad \sigma \in C - L,$$

$$\arg(\sigma^\pm - e^{i\phi}) = \eta_2^*, \quad \sigma \in L.$$ (4.14)

By the definition given in (4.13), (4.14), we find that when a point moves across the boundary, from $S^-$ to $S^+$, the term $(\eta_1 + \eta_2)$ in (4.12) may undergo a decrease $4\pi$ and $2\pi$ for $\sigma \in C - L$ and $\sigma \in L$ respectively. Meanwhile, the term $(\eta_1 - \eta_2)$ in (4.12) may undergo a decrease $0$ and $2\pi$ for $\sigma \in C - L$ and $\sigma \in L$ respectively. Thus by (4.12), the relation between $\tilde{\Gamma}(\sigma^+)$ and $\tilde{\Gamma}(\sigma^-)$ can be represented as

$$\tilde{\Gamma}(\sigma^+) = \tilde{\Gamma}(\sigma^-), \quad \sigma \in C - L,$$

$$\tilde{\Gamma}(\sigma^+) = \ll -e^{-2\pi \epsilon_\alpha} \gg \tilde{\Gamma}(\sigma^-), \quad \sigma \in L.$$ (4.15)
Note that the term $\ll -e^{-2\pi \alpha} \gg$ representing the jump when $\Gamma(\zeta)$ moves across the segment $L$ of the boundary is exactly the factor $e^{2\pi i \delta}$ of the eigenrelation (2.38). This implies that the definition given in (4.13) and (4.14) will provide us the required values for the function $\Gamma(\zeta)$.

For a point $\zeta$ not on the boundary and $\text{arg}(\zeta) = \psi$, we use $\eta_1^*$ as a reference to calculate the argument $\eta_1 = \text{arg}(\zeta - e^{-i\psi})$, i.e.

$$\eta_1 = \eta_1^* + \Delta \eta,$$  \hspace{1cm} (4.16a)

where

$$\Delta \eta = \text{arg}\left(\frac{\zeta - e^{-i\psi}}{\sigma - e^{-i\psi}}\right).$$  \hspace{1cm} (4.16b)

$\eta_2 = \text{arg}(\zeta - e^{i\phi})$ can be obtained in a similar way. Thus, the multi-valued function $\Gamma(\zeta)$ can be uniquely determined by (4.12) throughout the whole body except the hole.

One may ask why not calculate the argument $\eta_1 = \text{arg}(\zeta - e^{-i\phi})$ and $\eta_2 = \text{arg}(\zeta - e^{i\phi})$ by the usual way used in the calculation of the polar coordinates. The reason is that if we calculate the arguments in such a way, it will cause a difference $2\pi$ when $\zeta$ moves across the r-axis shown in Figures 5a,b. However, there should not exist any discontinuity across the r-axis. Hence, it is not appropriate to calculate the arguments $\eta_1$ and $\eta_2$ by the usual way.

By the algorithm suggested above, the hoop stress in nondimensionalized form, i.e. $\sigma_{nn}/(\hat{p}/2b \sin \phi)$, for orthotropic materials is depicted in Figure 6 with the following physical information $a = 1.0 \text{ m}$, $b = 0.6 \text{ m}$, $\phi = 30^\circ$, $\hat{\zeta} = (\hat{p},0,0)^T$ where $\hat{p} = 1000.0 \text{ Nt}$. Moreover, the contour plots of nondimensionalized stress $\sigma_{11}, \sigma_{12}$ and $\sigma_{22}$ are also depicted in Figure 7, 8 and 9. From these figures, it can be observed clearly that the stress singularities occurs at the ends of contact region, and the stress distribution is symmetric, which is due to the feature of the present considered problem.
CHAPTER 5
CONCLUSIONS

By applying Stroh's formalism and the method of analytical continuation, a systematic formulation is established to solve the mixed boundary value problems of two-dimensional anisotropic elastic bodies with elliptical boundaries. The generality of our solutions is shown as follows. First, the considered bodies are general anisotropic media, and the obtained solutions can be reduced to those for orthotropic and isotropic media, e.g. the analytical solutions shown in Section 4.3. Second, the geometry of the elliptical hole include the special cases of circles and cracks. Moreover, the number of stamps applied on the hole boundary and their locations are arbitrary. Third, the stress and displacement boundary value problems may be considered as a limiting case of the mixed boundary value problems, i.e., the solutions of the stress and displacement boundary value problems can be deduced from the present solutions. Last, although we deal with two-dimensional problems, the in-plane displacements and stresses as well as the anti-plane displacement $u_3$ and stresses $\sigma_3$ are both considered in the present formulation.

Besides the generality of the solutions stated in the last paragraph, the most important result is that we establish a systematic approach to deal with the mixed boundary value problems for two-dimensional anisotropic elasticity. We hope that all the other unsolved problems may be solved by a similar way. The solution procedure is as follows.

1. Equations (2.9a,b,c): General solution for two-dimensional anisotropic elasticity, which satisfies the kinematic equations, Hooke's law and equilibrium conditions. The only unknown in equations (2.9a,b) is the complex function vector $f(z)$.

2. Equations (2.22a,b): Set up the boundary conditions for the mixed boundary value problems through the physical requirements.
3. **Equation (2.31):** Transform the above boundary conditions into a *Hilbert problem* by the method of analytical continuation, in which $\tilde{\psi}(z)$ should be related to the unknown function $f(z)$.

4. **Equation (2.34):** *Solve* the Hilbert problem completely through assistance of the physical conditions such as infinity behavior, singular behavior and resultant force equilibrium, etc.

5. **Equations (2.12a,b), (3.12), (3.13), (4.11):** *Special treatment* of the solutions, such as the explicit full field domain solution, stresses under the punch, the displacement gradient outside the punch, hoop stress around the hole boundary, reduction to the special cases like isotropic media, · · · etc.

In application, the solutions obtained here can provide a more clear understanding of the effects induced by each variable involved in the considered problems, especially for the anisotropic case. In engineering practice, since there are many important structures composed of anisotropic materials such as composite laminates, which are widely used in aeronautical and astronautical engineering due to their high strength-weight ratio, it is important to know the behavior of these materials under various loading conditions. Our solutions can provide an alternative view which is more precise than those provided by numerical methods. Furthermore, the solutions derived here may be employed as a basis to develop a more effective numerical method such as boundary element method or boundary collocation method, or to solve another unsolved elasticity problems.
REFERENCES


Figure 1  A multiply connected region with interior boundaries
Figure 2  A lacet $C_k$ surrounding an arc $L_k$
Figure 3  An infinite medium containing an elliptical hole
Figure 4  The hole problems in the $\zeta$-domain.
Figure 5  A special polar coordinate system for the calculation of the multi-valued function $\Sigma(\zeta)$
Figure 6 The nondimensionalized hoop stress $\frac{\sigma_{nn}}{(\bar{p}/2b\sin\phi)}$ diagram for the elliptical hole problem in an orthotropic medium.
Figure 7  Contour plot of the nondimensionalized stress $\sigma_{11}/(\bar{p}/2b\sin \phi)$ for the elliptical hole problem in an orthotropic medium.
Figure 8  Contour plot of the nondimensionalized stress $\sigma_{12}/(\bar{p}/2b\sin\phi)$ for the elliptical hole problem in an orthotropic medium.
Figure 9  Contour plot of the nondimensionalized stress $\sigma_{22}/(\bar{p}/2b \sin \phi)$ for the elliptical hole problem in an orthotropic medium.