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MIXED BOUNDARY VALUE PROBLEMS FOR
ANISOTROPIC ELASTICITY

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ABSTRACT

By combining Stroh's formalism and the method of analytical continuation, several mixed-typed boundary value problems of an anisotropic elastic half-plane are studied in this report. Firstly, we consider a set of rigid punches of arbitrary profiles indenting into the surface of an anisotropic elastic half-plane with no slip occurring. Illustrations are presented for the normal and rotary indentation by a flat-ended punch. A sliding punch with or without friction is then considered under the complete or incomplete indentation condition.

Keywords: Anisotropic Elasticity, Mixed Boundary Value Problem, Stroh Formalism, Punch Indentation
CONTENTS

Acknowledgements ................................................................. i
Abstract ................................................................................... ii
Contents ................................................................................... iii
List of Figures ........................................................................... iv
1. Introduction ............................................................................. 1-3
2. Anisotropic Elastic Half-Plane ............................................. 4-8
   2.1 Stroh's Formalism and Analytical Continuation ................... 4-6
   2.2 Semi-Infinity Condition ..................................................... 6-8
3. First Type of Mixed Boundary Value Problems
   – Punch Problems (No Slip) .................................................... 9-15
   3.1 Indentation by a Flat-Ended Punch ..................................... 11-14
   3.2 A Flat-Ended Punch Tilted by a Couple .............................. 14-15
4. Second Type of Mixed Boundary Value Problems
   – A Sliding Punch With Friction ............................................. 16-20
5 Conclusions ............................................................................... 21-21
References .................................................................................. 22-24
Appendix A.................................................................................. 25-26
Appendix B .................................................................................. 27-28
Figures ....................................................................................... 29-31
List of Figures

Figure 1: Notation of the half-plane

Figure 2: A wedge-shaped punch under normal pressure

Figure 3: A lacet integral contour
CHAPTER 1
INTRODUCTION

The problem of plane punch indentation has been investigated for many years due to its broad application in engineering mechanics. This is one of the mixed boundary value problems and may be considered as a particular contact problem because of the line contact region. For the contact problems, most of the analytical formula can be found in the books written by Galin (1961), Gladwell (1980) and Johnson (1985). Muskhelishvili (1954) and England (1971) provided solutions for several types of punch problems in their books by using the method of analytical continuation. Gladwell and England (1977) and Gladwell (1978) have investigated the use of certain orthogonal polynomial expansions in the solutions of some mixed boundary value problems such as crack and punch problems. Frictional punch of flat-ended or wedge-shaped profile with crack initiating at one end of the contact region has been studied by Hasebe et al. (1989) and Okumura et al. (1990) who used a rational mapping function and complex stress function to carry out the analysis. Fabrickant (1986a,b) presented an integral equation based on the reciprocal distance established by himself to solve the problem for a punch of arbitrary shape on an elastic half-space. A similar case for an elliptical punch on an elastic half-space with friction was analyzed by Shibuya et al. (1989) who used the generalized Abel transform method.

The literature survey stated in the above paragraph is for the cases of isotropic materials. For anisotropic materials, Willis (1966) studied the Hertzian contact problem of anisotropic bodies by the Fourier transform method. Chen (1969) investigated stresses fields in anisotropic half-plane due to indentation and sliding by a frictionless punch with smooth end face. Tsiang and Mandell (1985) employed a two-dimensional assumed stress hybrid finite element to obtain the characteristic matrices of the bodies brought into con-
tact. Shield (1987) provided the variational principles for some elastic problems involving smooth contact and crack problem. Jaffar and Savage (1988) investigated the contact problem in which an elastic strip is indented by a rigid punch of arbitrary shape by using a numerical method proposed by Gladwell (1976). Klintworth and Stronge (1990) used a potential function approach to construct solutions for planar punch problems in an anisotropic half-plane where there is no slip on the surface of a flat punch. In contrast to homogeneous materials, a multi-layered medium bounded to an elastic half-space has also been examined by many researchers, for example, recently Lin et al. (1991) applied the complex potential functions suggested by Green and Zerna (1954) and Pan and Chou (1976) to obtain the closed form solution for a transversely isotropic half-space subjected to various distribution of normal and tangential contact stresses on its surface, and Kuo and Keer (1992) used the Hankel transform to numerically solve the contact problem of a layered transversely isotropic half-space.

Although there are many results obtained in the literature, to the authors’ knowledge the analytical solutions for the general punch problems, such as a problem of arbitrary number of punches with arbitrary profiles, have not been found for the general anisotropic elastic half-plane. If such a general solution can be found analytically, the effects of material anisotropy and punch profiles can be studied easily, which is helpful for the physical applications to composite materials, road pavements, geotechnical engineering and tribology, etc. In the present paper, Stroh’s formalism (Stroh, 1958, 1962) which has been proved to be elegant and powerful for anisotropic elasticity are combined with the method of analytical continuation (Muskhelishvili, 1954) to solve the problems of indentation of plane punches with arbitrary profiles into an anisotropic elastic half-plane. Explicit solutions expressed in complex matrix notation are obtained from the Hilbert problem of vector form
(Muskhelishvili, 1954; Hwu, 1992). For the purpose of illustration, some special cases are deduced from this general solution, such as normal and rotary indentation by a flat-ended punch. Moreover, in order to verify our results, the solutions are simplified to the cases of isotropic materials and the results agree with those given by Muskhelishvili (1954). The other case which has also been studied in this paper is the problem of a sliding punch with friction. An example of incomplete indentation by a wedge-shaped punch under normal pressure is solved explicitly and the condition to have complete indentation is discussed.
CHAPTER 2
ANISOTROPIC ELASTIC HALF-PLANE

2.1 Stroh’s Formalism and Analytical Continuation

The basic equations for two-dimensional anisotropic elasticity are the strain-displacement equations, the stress-strain laws and the equations of equilibrium. By applying the Stroh’s formalism (Stroh, 1958, 1962), a general solution satisfying these equations may be expressed as

$$\tilde{u} = \tilde{A} \tilde{f}(z) + \overline{\tilde{A} \tilde{f}(z)}, \quad \tilde{\phi} = \tilde{B} \tilde{f}(z) + \overline{\tilde{B} \tilde{f}(z)}, \quad (2.1a)$$

where

$$\tilde{A} = \begin{bmatrix} a_1 & a_2 & a_3 \\ \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} b_1 & b_2 & b_3 \\ \end{bmatrix},$$

$$\tilde{f}(z) = \begin{bmatrix} f_1(z_1) \\ f_2(z_2) \\ f_3(z_3) \end{bmatrix}, \quad z_\alpha = x + p_\alpha y, \quad \alpha = 1, 2, 3. \quad (2.1b)$$

In the above equation, \((x, y)\) is a fixed rectangular coordinate system; \(\tilde{u}\) and \(\tilde{\phi}\) represent, respectively, the displacements and stress functions; \(p_\alpha, (a_\alpha, b_\alpha), \alpha = 1, 2, 3\), are the eigenvalues and eigenvectors of the materials; \(\tilde{f}(z)\) is a holomorphic complex function vector to be determined by satisfying the boundary condition of the problems considered. The superscript \(T\) denotes the transpose and the overbar represents the conjugate of a complex number. Note that (Suo, 1990, Hwu, 1993) in the derivation throughout this paper, the argument of each component function of \(\tilde{f}(z)\) is written as \(z = x + py\) without referring to the associated eigenvalues \(p_\alpha\). Once the solution of \(\tilde{f}(z)\) is obtained for a given boundary value problem, a replacement of \(z_1, z_2\) or \(z_3\) should be made for each component function to calculate field quantities from (2.1).

The stresses \(\sigma_{ij}\) are related to the stress function \(\tilde{\phi}\) by

$$\sigma_{i1} = -\tilde{\phi}_{i,2}, \quad \sigma_{i2} = \tilde{\phi}_{i,1}. \quad (2.2a)$$
More generally, if $\tilde{t}$ is the surface traction vector, then

$$
\tilde{t} = \frac{d \varphi}{ds},
$$

(2.2b)

where $s$ is the arc length measured along the curved boundary in the direction such that when one faces the direction of increasing $s$, the material is located on the right hand side.

Through the use of the analytical continuation method, a Hilbert problem (Muskhe-lishvili, 1954) can be formulated for the half-plane problems. It will be assumed that the elastic body occupies the lower half-plane $y < 0$ which is denoted by $S^-$, so that the region $S^-$ is to the right if one moves in the positive direction along the $x$-axis. The upper half-plane is denoted by $S^+$ (Fig.1).

Since $ds$ is equal to $dz$ on $y = 0$, the traction on the surface $y = 0$ of the half-plane $S^-$ can be represented as

$$
\tilde{t} = \lim_{y \to 0^-} \frac{d}{dz} \left\{ \widetilde{B} \tilde{f}(z) + \overline{\widetilde{B}} \overline{\tilde{f}(z)} \right\},
$$

(2.3)

and the last term of (2.3) has the following relation

$$
\lim_{y \to 0^-} \frac{d}{dz} \overline{\widetilde{B}} \overline{\tilde{f}(z)} = \lim_{y \to 0^+} \frac{d}{dz} \overline{\widetilde{B}} \overline{\tilde{f}(z)}.
$$

From the complex theory, we know that if $\tilde{f}(z)$ is holomorphic for $z \in S^-$, then $\overline{\tilde{f}(z)}$ is holomorphic for $z \in S^+$, and $\frac{d}{dz} \overline{\tilde{f}(z)} = \overline{\tilde{f}'(z)}$ where prime (') denotes differentiation with respect to its argument. With this in mind, we introduce $\vartheta'(z)$ such that

$$
\vartheta'(z) = \begin{cases} 
\widetilde{B} \tilde{f}'(z), & z \in S^-, \\
-\overline{\widetilde{B}} \overline{\tilde{f}'(z)}, & z \in S^+.
\end{cases}
$$

(2.4)

Since $\tilde{f}'(z)$ should be holomorphic in the elastic body $S^-$, $\vartheta'(z)$ is now holomorphic in $S^-$ and $S^+$, i.e. $\vartheta'(z)$ is sectionally holomorphic in the whole plane except possibly on some segments of $x$-axis.
By the above definition and the following notation,

\[
\lim_{y \to 0^-} \tilde{\theta}'(z) = \tilde{\theta}'(x^-), \quad \lim_{y \to 0^+} \tilde{\theta}'(z) = \tilde{\theta}'(x^+),
\]

(2.3) can be rewritten as

\[
\tilde{t}(x) = \lim_{y \to 0^-} B \tilde{\theta}'(z) + \lim_{y \to 0^+} B \tilde{\theta}'(\bar{z}) = \tilde{\theta}'(x^-) - \tilde{\theta}'(x^+).
\]

(2.5)

In a similar way, the differentiation of displacement vector \( \tilde{u} \) shown in (2.1a) for \( y \to 0^- \) can be written as

\[
i M \tilde{u}'(x) = \tilde{\theta}'(x^+) + \bar{M}M^{-1} \tilde{\theta}'(x^-).
\]

(2.6)

where \( M = -iB\bar{A}^{-1} = H^{-1}(I + iS) = L(I - iS)^{-1} \) is the impedance matrix, and \( S, H \) and \( L \) are three real fundamental elasticity matrices (Ting, 1988).

2.2 Semi-Infinity Condition

Consider an arc \( ab \) in the body \( S^- \) having the direction from \( a \) to \( b \) as its positive direction (Fig.1). By using (2.2b) and (2.1a)\( _2 \), the resultant force \( q \) on the arc \( ab \) can be represented as

\[
q = \int_a^b \tilde{t} ds = [B \tilde{\theta}(z) + \bar{B} \tilde{\theta}(\bar{z})]_a^b.
\]

(2.7)

If we consider the case where the stresses and rotation tend to zero as \( |z| \) tends to infinity, for large \( |z| \) the complex function vector \( \tilde{f}(z) \) has the form

\[
\tilde{f}(z) = \langle \log z_\alpha \rangle q^* + O(1),
\]

(2.8)

where the angular bracket \( \langle \rangle \) stands for the diagonal matrix in which each component is varied according to the Greek index \( \alpha \), \( q^* \) is a complex constant vector to be determined.
by the semi-infinity condition. With the arc \( ab \) lying on the boundary of the half-plane, i.e. \( x \)-axis, we let \( a = R_1 e^{i\pi} \), \( b = R_2 e^{i2\pi} \). The resultant force \( \varrho \) applied on the surface of the half-plane can now be calculated by substituting (2.8) into (2.7). The result is

\[
\varrho = (B \varrho^* + \overline{B} \overline{\varrho}^*) \log \frac{R_2}{R_1} + \pi i (B \varrho^* - \overline{B} \overline{\varrho}^*).
\]

Since \( R_1, R_2 \) tend to infinity independently, we must conclude

\[
B \varrho^* + \overline{B} \overline{\varrho}^* = 0,
\]

\[
\varrho = \pi i (B \varrho^* - \overline{B} \overline{\varrho}^*),
\]

and hence

\[
\varrho^* = \frac{1}{2\pi i} \overline{B}^{-1} \varrho.
\]

By (2.4), (2.8) and (2.10), the semi-infinity condition for \( \varrho' (z) \) is

\[
\varrho' (z) = \frac{1}{2\pi i} \overline{B} \ll \frac{1}{z_\alpha} \gg \overline{B}^{-1} \varrho, \quad \text{as } |z| \to \infty.
\]

If \( y \to 0^- \), the diagonal matrix \( \ll \frac{1}{z_\alpha} \gg \) approximates \( \frac{1}{x} I \overline{\sim} \) since the second part of \( z_\alpha \), \( p_\alpha y \), disappears. Therefore

\[
\varrho' (x) = \frac{\varrho}{2\pi i x}, \quad \text{as } |x| \to \infty \text{ and } y \to 0^-.
\]

In summary, instead of finding \( f_{\varrho} (z) \) in \( S^- \), we define a new function \( \varrho' (z) \) which is sectionally holomorphic in the whole plane except on some segments of the boundary, and by solving (2.5) and/or (2.6) with the semi-infinity condition derived in (2.12), \( \varrho' (z) \), hence \( f_{\varrho} (z) \), can be determined for the half-plane problem.

One thing that should be emphasized here is the applicability of \( f_{\varrho} (z) \) in the full field of the half-plane. As we know, the general solution shown in (2.1) requires that each component of the complex function vector \( f_{\varrho} (z) \) be a holomorphic function with argument

7
$z_1$, $z_2$ and $z_3$ respectively. This means that each component of the new function $\tilde{\theta}'(z)$ introduced in (2.4) may not be a function with only one argument $z_1$ or $z_2$ or $z_3$. The question now is that $\tilde{\varphi}(z)$ is determined after $\tilde{\theta}'(z)$ is found through a certain Hilbert problem and how can we know the argument of $\tilde{\theta}'(z)$ is $z_1$, $z_2$ or $z_3$ or any combination of them. In solving the Hilbert problem, which is set through the method of analytical continuation for a specific boundary, the full field solution may be expressed in terms of the complex variable $z = x + iy$. In the case of half-plane problem, this specific boundary is $x$–axis in which $y = 0$, hence the full field solution may be expressed in terms of $z$ or $z_\alpha(= x + p_\alpha y)$, $\alpha = 1, 2, 3$, depending on the requirement of $\tilde{\varphi}(z)$. Therefore, the most appropriate way to get the solution, which is valid for the full field including the boundary, is dropping the subscripts of $z$ during the derivation of $\tilde{\theta}'(z)$ and $\tilde{\varphi}(z)$. Once the solution of $\tilde{\varphi}(z)$ is obtained from a certain boundary value problem, a replacement of $z_1$, $z_2$ or $z_3$ should be made for each component function to calculate the full field solution for the deformations and stresses. A translating technique for the above statement has been introduced by Hwu (1993) as follows.

If an implicit solution is written as

$$f(z) = \mathcal{C} \ll g_\alpha(z) \gg g, \quad (2.13a)$$

with the understanding that the subscript of $z$ is dropped before matrix product and a replacement of $z_1$, $z_2$ or $z_3$ should be made for each component function of $f(z)$ after the multiplication of matrices, the explicit solution can be expressed as

$$f(z) = \sum_{k=1}^{3} \ll g_k(z_\alpha) \gg \mathcal{C}_k g, \quad (2.13b)$$

where

$$\mathcal{I}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{I}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{I}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.13c)$$
CHAPTER 3
FIRST TYPE OF MIXED BOUNDARY VALUE PROBLEMS
- PUNCH PROBLEMS (NO SLIP)

In the following sections, a variety of mixed boundary value problems for the half-plane $S^-$ will be considered. In all cases, the analytical continuation method described in section 2 will be used to represent the stress and displacement fields in terms of a single complex function vector which may be determined by satisfying the Hilbert problem for a set of line intervals on $y = 0$.

In this section, we examine the case that a set of rigid punches of given profiles are brought into contact with the surface of the half-plane and are allowed to indent the surface in such a way that the punches completely adhere to the half-plane on initial contact and during the subsequent indentation no slip occurs and the contact region does not change. Let us suppose the contact region $L$ is the union of a finite set of line segments $L_k = (a_k, b_k), k = 1, 2, \ldots, n$, where the ends of the segments are encountered in the order $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ when moving in the positive $x$-direction. For this case the displacements of the surface of the half-plane are known at each point of the contact region, then the boundary conditions are

\[ u(x) = (u_k(x), v_k(x) + c_k, 0)^T = \hat{u}(x), \quad x \in L, \]
\[ \hat{t}(x) = (\sigma_{xy}, \sigma_{yy}, \sigma_{zy})^T = \hat{\sigma}, \quad x \notin L, \]

where $u_k(x)$ and $v_k(x)$ are related to the profile of the $k$th punch and $c_k$ is the relative depth of indentation. From (2.5) and (2.6), the boundary conditions lead to the following Hilbert problem

\[ \hat{\theta}'(x^+) - \hat{\theta}'(x^-) = \hat{0}, \quad x \notin L, \]
\[ \hat{\theta}'(x^+) + \overline{MM}^{-1} \hat{\theta}'(x^-) = i\overline{M} \hat{\mu}'(x), \quad x \in L. \]

9
The solution to this Hilbert problem of vector form is (Hwu, 1992)

\[ \theta'(z) = \frac{1}{2\pi} \sum_0(z) \int L \frac{1}{t-z} [\sum_0^+(t)]^{-1} \sum_\Lambda \sum(i) dt + \sum_0(z) \sum_n(z), \]  

(3.3a)

where \( \sum_n(z) \) is an arbitrary polynomial vector with degree not higher than \( n \), and \( \sum_0(z) \) is the basic Plemelj function satisfying

\[ \sum_0^+(x) = \sum_0^-(x), \quad x \notin L, \]  

(3.3b)

\[ \sum_0^+(x) + \sum M^{-1} \sum_0^-(x) = 0, \quad x \in L, \]

i.e.

\[ \sum_0(z) = \sum \Gamma(z), \]  

(3.3c)

where

\[ \Lambda = [\lambda_1, \lambda_2, \lambda_3], \]

\[ \Gamma(z) = \prod_{j=1}^n (z - a_j)^{-1+\delta_j} (z - b_j)^{\delta_j}, \]  

(3.3d)

\( \delta_\alpha \) and \( \lambda_\alpha, \alpha = 1, 2, 3 \) of (3.3d) are the eigenvalues and eigenvectors of

\[ (\sum M^{-1} + e^{2\pi i \delta} \sum M^{-1}) \Lambda = 0. \]  

(3.4a)

The explicit solutions for the eigenvalues \( \delta \) are (see Appendix A)

\[ \delta_\alpha = -\frac{1}{2} + i \epsilon_\alpha, \quad \alpha = 1, 2, 3, \]  

(3.4b)

where

\[ \epsilon_1 = \epsilon = \frac{1}{2\pi} \ln \frac{1 + \beta}{1 - \beta}, \quad \epsilon_2 = -\epsilon, \quad \epsilon_3 = 0, \]

(3.4c)

\[ \beta = [-\frac{1}{2} tr(\sum^2)]^{1/2}, \quad \sum = i(\sum \sum^T - \Lambda), \]

\( tr \) stands for the trace of matrix. Moreover, for normalizing the eigenvector matrix \( \Lambda \), the normalization proposed by Hwu (1993) may be slightly changed to fit the present case, i.e.

\[ \frac{1}{2} \sum^T (\sum^{-1} + \sum^{-1}) \Lambda = \Lambda. \]  

(3.4d)
To determine \( \mathcal{P}_n(z) \), we see that it is at most a polynomial of degree \( n - 1 \)

\[
\mathcal{P}_n(z) = \tilde{d}_0 + \tilde{d}_1 z + \cdots + \tilde{d}_{n-1} z^{n-1}.
\]  

(3.5)

and also from (2.12)

\[
\tilde{d}_{n-1} = \frac{1}{2\pi i} \Lambda^{-1} \mathcal{Q}.
\]

(3.6)

As to the remaining \( (n - 1) \) unknown coefficients of \( \mathcal{P}_n(z) \), additional physical assumptions are required before the problem is solved completely. Let us suppose that the resultant forces applied to each punch are known. Then if \( \mathcal{Q}_k \) is the known resultant force vector on \( L_k \), we find from (2.4) and (2.7)

\[
\mathcal{Q}_k = -\int_{L_k} [\mathcal{Q}'(x^+) - \mathcal{Q}'(x^-)] dx,
\]

(3.7)

for \( k = 1, 2, \ldots, n \). Substituting (3.3a) into (3.7) yields \( n \) equations for the determination of the \( n \) coefficient vectors \( \tilde{d}_j \). It is apparent that one of these equations is redundant as (3.6) ensures the overall equilibrium of the elastic body with

\[
\mathcal{Q} = \sum_{k=1}^{n} \mathcal{Q}_k.
\]

However these \( n \) equations completely determine the solution (3.3a).

Now the problem is solved in principle. For illustrating the solutions derived above, some special cases will be investigated as follows.

3.1 Indentation by a Flat-ended Punch

We first examine the case of indentation by a single punch with a flat-ended profile which makes contact with \( S^- \) over the region \( |x| \leq a \), and the force \( \mathcal{Q} \) applied on the punch is given. Then

\[
\tilde{u}'(x) = \tilde{0},
\]

(3.8)
and from (3.3a),(3.3c) and (3.6) we find

\[ \tilde{\theta}'(z) = \frac{1}{2\pi i} \tilde{\eta}(z) \tilde{\Lambda}^{-1} \tilde{g}, \]  

(3.9)

where

\[ \tilde{\eta}(z) = \ll \frac{1}{\sqrt{z^2 - a^2}} \left( \frac{z + a}{z - a} \right)^{-i\epsilon} \gg. \]

The stresses under the punch can then be determined by using (2.5), (3.2)_2 and (3.8), i.e.

\[ \tilde{\tau}(x) = (\tilde{I} + \overline{M} M^{-1}) \tilde{\theta}'(x^-), \quad |x| \leq a. \]  

(3.10)

Since the stresses \( \tilde{\tau}(x) \) are real, the result of the right hand side of (3.10) manipulated by some complex matrices should be real. Therefore it is of interest to obtain the real form of the solution, because it should provide a better understanding of the physical behaviour of the stress field under the punch. To this end, the following equalities derived in a way similar to that presented in the Hwu's paper (1993) are used for the simplification of (3.10), i.e.

\[ (\tilde{I} + \overline{M} M^{-1}) \tilde{\Lambda} = 2\Lambda \ll e^{-\pi \epsilon \cosh(\pi \epsilon)} \gg, \]  

(3.11)

and

\[ \ll \Lambda \gg \ll \Lambda^{-1} \gg = \tilde{I} + \frac{1 - c_R}{\beta^2} \tilde{S}^{T2} + \frac{c_I}{\beta} \tilde{S}, \]  

(3.12)

where \( c_1 = c, \ c_2 = \bar{c}, \ c_3 = 1 \) and \( c_R, \ c_I \) are real and imaginary parts of \( c \) which is an arbitrary complex number. Also, for \( y \to 0^- \) and \( |x| \leq a \), it can be shown that

\[ \tilde{\eta}(x^-) = \ll \frac{i e^{\pi \epsilon}}{\sqrt{a^2 - x^2}} e^{-i \epsilon \ln \frac{a + x}{a - x}} \gg. \]  

(3.13)

By the use of (3.9) and (3.11) – (3.13), equation (3.10) can be written in real form as

\[ \tilde{\tau}(x) = \frac{1}{\pi \sqrt{a^2 - x^2}} [\tilde{I} + \frac{1 - c_R}{\beta^2} \tilde{S}^{T2} + \frac{c_I}{\beta} \tilde{S}] \tilde{g}, \quad |x| \leq a, \]  

(3.14)
where
\[ c_R + ic_I = \cosh(\pi \varepsilon) e^{-i\varepsilon \ln \frac{1 + \bar{\gamma}}{1 + \bar{\gamma}}}. \]

For orthotropic materials, (3.12) can be expressed as
\[
\Lambda \ll c_\alpha \gg \Lambda^{-1} = \begin{bmatrix}
c_R & c_I \sqrt{-\frac{S_{21}}{S_{12}}} & 0 \\
-c_I \sqrt{-\frac{S_{12}}{S_{21}}} & c_R & 0 \\
0 & 0 & 1
\end{bmatrix}, \tag{3.15}
\]

where \( S_{12}, S_{21} \) are the \{12\} and \{21\} components of \( S \). Then \( \tilde{t}(x) \) becomes
\[
\tilde{t}(x) = \frac{1}{\pi \sqrt{a^2 - x^2} \sqrt{1 - \beta^2}} \begin{bmatrix}
\cos(\varepsilon \ln \frac{a + x}{a - x}) q_x - \sqrt{-\frac{S_{12}}{S_{21}}} \sin(\varepsilon \ln \frac{a + x}{a - x}) q_y \\
\sqrt{-\frac{S_{12}}{S_{21}}} \sin(\varepsilon \ln \frac{a + x}{a - x}) q_x + \cos(\varepsilon \ln \frac{a + x}{a - x}) q_y \\
q_z
\end{bmatrix}, \quad |x| \leq a, \tag{3.16}
\]

where \( q_x, q_y \), and \( q_z \) are the components of the force vector \( \mathbf{q} \).

It should be noted that the general solutions shown in section 2 are valid only for the nondegenerate materials, that is, the material eigenvalues \( p_\alpha, \alpha = 1, 2, 3 \), are distinct or three independent material eigenvectors \( a_\alpha, b_\alpha, \alpha = 1, 2, 3 \), can be found when \( p_\alpha \) are repeated. Otherwise, the general solutions shown in section 2 should be modified (Ting, 1982). However, if the final solutions do not contain any material eigenvalues \( p_\alpha \) or eigenvectors \( A, B \) explicitly, and are composed of the real fundamental elasticity matrices such as \( S, H, \mathcal{L} \) and \( N_i \) (Ting, 1988), they may be applied to any kind of anisotropic materials including the degenerate materials such as the isotropic materials. Following is the presentation for the reduction to isotropic materials.

Consider the special case of isotropic body and the force \( \mathbf{q} \) applied on the punch is given as \((0, -q_0, 0)^T\). Knowing that for isotropic body
\[
S = \frac{\kappa - 1}{\kappa + 1} \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

13
where \( \kappa = 3 - 4\nu \) for plane strain conditions and \( \kappa = \frac{3-\nu}{1+\nu} \) for generalized plane stress condition, and \( \nu \) is the Poisson’s ratio, we have by (3.4c)

\[
\beta = \frac{\kappa - 1}{\kappa + 1}, \quad \epsilon = \frac{1}{2\pi} \ln \kappa. \tag{3.17}
\]

From (3.16) and (3.17), the stresses under the punch can then be found explicitly as

\[
\mathbf{t}(x) = \begin{cases} 
\sigma_{xy} & = \frac{1 + \kappa}{\sqrt{\kappa}} \frac{q_0}{2\pi \sqrt{a^2 - x^2}} \left\{ \begin{array}{c}
\sin \left( \epsilon \ln \frac{a+x}{a-x} \right) \\
-\cos \left( \epsilon \ln \frac{a+x}{a-x} \right)
\end{array} \right. \\
\sigma_{yy} & = 0 \\
\sigma_{xy} & = 0
\end{cases}, \tag{3.18}
\]

which agree with those shown in Muskhelishvili (1954).

### 3.2 A Flat-ended Punch Tilted by a Couple

A second problem illustrating the above theory is the case of a flat-ended punch which adheres to the half-plane \( S^- \) and is then tilted by the application of a couple \( m \). Let us suppose the punch is of width \( 2a \) and is tilted through a small angle \( \epsilon \) measured in the counterclockwise direction. Then in (3.2)

\[
\mathbf{u}'(x) = \epsilon \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \epsilon \mathbf{\hat{z}}_2, \quad |x| \leq a, \tag{3.19}
\]

and from (3.3a)

\[
\theta'(z) = \frac{\epsilon}{2\pi} \mathcal{X}_0(z) \int_{-a}^{a} \frac{1}{t - z} [\mathcal{X}_0^+(t)]^{-1} dt \mathcal{M}_{\mathbf{\hat{z}}_2}. \tag{3.20}
\]

Note that the last term in (3.3a) vanishes since the resultant forces are zero. In order to evaluate the above integral of vector form, a special technique similar to that presented in the book of England (1971) for line integrals of scalar form has been developed in Appendix B. Applying that technique we find

\[
\theta'(z) = i\epsilon \left\{ \int_{-a}^{a} \mathcal{X}_0(z) \ll z + 2ia\epsilon \gg \Lambda_{\mathbf{\hat{z}}_2}^{-1} \right\} \left( \int_{-a}^{a} \mathcal{M}_{\mathbf{\hat{z}}_2} \mathcal{M}_{\mathbf{\hat{z}}_2}^{-1} \right) \mathcal{M}_{\mathbf{\hat{z}}_2}. \tag{3.21}
\]

This now enables us to calculate the stresses over the contact region in terms of the angle of tilt \( \epsilon \). However, in an alternative problem it may be assumed that the couple \( m \) acting
on the punch is given and it is required to find the corresponding angle of tilt \( \varepsilon \). Hence it is necessary to evaluate the relation between the applied couple \( m \) and the angle of tilt \( \varepsilon \).

For this purpose, we first calculate the stresses under the punch, by (2.5), (3.2)_2, (3.19) and (3.4d), we have

\[
\tilde{t}(x) = -\frac{i}{2} \varepsilon (I + \bar{M}M^{-1}) \tilde{\Lambda} \tilde{\Gamma}(x^-) \ll \varepsilon + 2ia\varepsilon_\alpha \gg \bar{\Lambda}^T \tilde{\iota}_2, \quad |x| \leq a. \quad (3.22)
\]

With this result, the couple \( m \) may now be calculated by

\[
m = \int_{-a}^{a} x \sigma_{yy} dx = \int_{-a}^{a} x \tilde{i}_2^T \tilde{t} dx, \quad (3.23)
\]

in which the integral may be evaluated by a way similar to that presented in Appendix B. The result is

\[
m = \frac{\pi a^2 \varepsilon \tilde{i}_2^T \tilde{\Lambda} \ll 1 + 4\varepsilon_\alpha^2 \gg \bar{\Lambda}^T \tilde{\iota}_2. \quad (3.24)
\]

For a given couple \( m \), the angle of tilt \( \varepsilon \) is determined by

\[
\varepsilon = \frac{2m}{\pi a^2 \varepsilon \tilde{i}_2^T \tilde{\Lambda} \ll 1 + 4\varepsilon_\alpha^2 \gg \bar{\Lambda}^T \tilde{\iota}_2. \quad (3.25)
\]

Substituting this value of \( \varepsilon \) into (3.22), one obtains the stresses under the punch tilted by a given couple \( m \).
CHAPTER 4
SECOND TYPE OF MIXED BOUNDARY VALUE PROBLEMS
- A SLIDING PUNCH WITH FRICTION

Using the analytical continuation method in the preceding sections, we may also solve the problems of a sliding punch with or without friction. Since frictionless problems may be covered by setting the friction coefficient to be zero in the friction problems, in this section consideration will be limited to the case where friction exists and the punch is on the verge of equilibrium. The boundary conditions for this kind of problems may be expressed as

\[
\begin{align*}
T(x) &= \tan \lambda P(x) \\
v(x) &= g(x) + \text{constant} \\
T(x) &= P(x) = 0,
\end{align*}
\]

where \( P(x) \) and \( T(x) \) are respectively the absolute values of pressure and tangential stress, \( \lambda \) is the angle of limiting friction for the punch and is a constant under the punch, \( g(x) \) is a given function for the profile of the punch. As before, suppose the elastic body occupies the lower half-plane \( y < 0 \), we have the relation \( T(x) = \sigma_{xy} \) if the punch is propelled from left to right and \( T(x) = -\sigma_{xy} \) if the punch is propelled from right to left. Moreover, to ensure contact \( P(x) = -\sigma_{yy} \) and the first equation of (4.1a) only holds provided \( \sigma_{yy} < 0 \) which must be checked when the solution is obtained.

By using (2.5), the relation between the pressure \( P(= -\sigma_{yy}) \) and the tangential stress \( T(= \pm \sigma_{xy}) \) shown in (4.1a)_1 may be expressed as

\[
\theta'_1(x^+) - \theta'_1(x^-) = \mp \tan \lambda [\theta'_2(x^+) - \theta'_2(x^-)], \quad x \in L,
\]

where \( \theta_1 \) and \( \theta_2 \) are, respectively, the first and second components of \( \vec{\theta} \). Rearrangement...
gives
\[
\lim_{y \to 0^+} \left[ \theta_1'(z) \pm (\tan \lambda) \theta_2'(z) \right] = \lim_{y \to 0^-} \left[ \theta_1'(z) \pm (\tan \lambda) \theta_2'(z) \right].
\]

Thus the function \( \theta_1'(z) \pm (\tan \lambda) \theta_2'(z) \) is holomorphic in the whole plane including the point at infinity and it tends to zero as \( |z| \to \infty \) from (2.11), hence by Liouville’s theorem one can conclude that
\[
\theta_1'(z) \pm (\tan \lambda) \theta_2'(z) = 0. \tag{4.3}
\]

The problem now reduces to determine a sectionally holomorphic scalar function \( \theta_2(z) \) (or \( \theta_1(z) \)) satisfying the displacement boundary condition of (4.1a)2. This condition can be expressed in terms of \( \theta_2 \) by employing (2.6) and (4.3) into (4.1a)2, as
\[
\frac{d}{dx} \theta_2(x^+) + \theta_2(x^-) = i \frac{d}{dx} g(x), \tag{4.6}
\]

where \( d = \mp m_{12}^* \tan \lambda + m_{22}^*, \) \( m_{12}^* \) and \( m_{22}^* \) are the \( \{12\} \) and \( \{22\} \) components of the matrix \( \mathcal{M}^{-1} \). Equation (4.6) is a standard Hilbert problem, the solution to it is (Muskhelishvili, 1954)
\[
\theta_2'(z) = \frac{X(z)}{2\pi d} \int_{-\infty}^{\infty} \frac{g(t)}{X^+(t)(t-z)} dt + X(z)p_n(z), \tag{4.7}
\]

where
\[
X(z) = \prod_{k=1}^{n} (z - a_k)^{-\delta} (z - b_k)^{\delta - 1}, \tag{4.8}
\]
\[
\delta = \frac{1}{2\pi} \arg \left( -\frac{d}{d} \right), \quad 0 \leq \delta < 1.
\]

\( p_n(z) \) is an arbitrary polynomial with degree not higher than \( n \) and arg stands for the argument of a complex number. Note that \( \delta \) is a real number and hence there are no oscillatory singularities in the solution.

The problem now is solved in principle. For the purpose of illustration, we examine an example of incomplete indentation where the region of contact is unknown and has to be determined by assuming the stresses are bounded at the ends of the contact region.
Consider a wedge-shaped punch under a total pressure $q_0$ which induces the contact with half-plane $S^-$ and the motion of the punch is to the left as shown in Fig.2. The profile of the punch can be expressed as $g(x) = \varepsilon x$ where the origin is taken so that the contact region is $-a \leq x \leq a$. Then from (4.8), $X(z) = (z + a)^{-\delta} (z - a)^{\delta - 1}$ and from (4.7) we see that the evaluation of $\theta'_2(z)$ depends on the integral

$$
\int_{-a}^{a} \frac{dt}{X^+(t)(t - z)},
$$

which has been evaluated by Muskhelishvili (1954) as

$$
\int_{-a}^{a} \frac{dt}{X^+(t)(t - z)} = \frac{2\pi i d}{d + d} \left\{ \frac{1}{X(z)} - [z + (2\delta - 1)a]X(z) \right\}.
$$

To determine the polynomial $p_n(z)$, using the semi-infinity condition (2.12) one obtains

$$
p_n(z) = \frac{i q_0}{2\pi}.
$$

Substituting (4.9), (4.10) with $g'(t) = \varepsilon$ into (4.7), $\theta'_2(z)$ becomes

$$
\theta'_2(z) = \frac{i \varepsilon}{d + d} \left\{ 1 - [z + (2\delta - 1)a]X(z) \right\} + \frac{i q_0}{2\pi} X(z).
$$

The pressure $P(x)$ under the punch can now be calculated by

$$
P(x) = -\sigma_{yy} = \theta'_2(x^+) - \theta'_2(x^-)
$$

$$
= -\frac{i \varepsilon}{d + d} \left[ x + (2\delta - 1)a \right] \left[ X^+(x) - X^-(x) \right] + \frac{i q_0}{2\pi} \left[ X^+(x) - X^-(x) \right], \quad |x| \leq a.
$$

By using the bipolar coordinates $z + a = R_1 e^{i \varphi_1}$, $z - a = R_2 e^{i \varphi_2}$, it will be seen that

$$
X^\pm(x) = \frac{e^{\pm i \pi \delta}}{(a + x)^{\delta} (a - x)^{1-\delta}}, \quad |x| \leq a.
$$

Hence the pressure $P(x)$ can be simplified as

$$
P(x) = \frac{\sin \pi \delta}{\pi (a + x)^{\delta} (a - x)^{1-\delta}} \left\{ q_0 - \frac{2 \pi \varepsilon}{(d + d)} \left[ x + (2\delta - 1)a \right] \right\}.
$$
The above expression is valid for general anisotropic half-plane. It can be shown that for isotropic bodies

\[ m_{12}^* = \frac{i(\kappa - 1)}{4\mu}, \quad m_{22}^* = \frac{1 + \kappa}{4\mu}, \]  

(4.13)

where \( \mu \) is the shear modulus. By using these values for the calculation of \( d \) and \( \delta \), we have

\[ d = \frac{1}{4\mu} \{1 + \kappa + i \tan \lambda(\kappa - 1)\}, \]

\[ \delta = 1 - \gamma, \quad \gamma = \frac{1}{2\pi} \arg\left\{-\frac{1 + \kappa e^{2i\lambda}}{\kappa + e^{2i\lambda}}\right\}, \]

and equation (4.12) can be proved to agree with that given in (England, 1971). If the motion of the punch is to the right, the same expression as (4.12) will be obtained except that

\[ d = \frac{1}{4\mu} \{1 + \kappa - i \tan \lambda(\kappa - 1)\}, \quad \delta = \gamma \]

As stated in the beginning of this section, to have a complete indentation, the applied force \( q_0 \) should be large enough that the end-face of the punch touches the half-plane, i.e., the pressure \( P(x) \) should be positive under the punch. By letting \( P(\pm a) > 0 \), we may find the minimum requirement for the applied force \( q_0 \) to reach complete indentation;

\[ q_0 \geq \frac{4\pi \varepsilon \delta}{d + \bar{d}} a. \]  

(4.14)

However, if \( q_0 \) is not sufficiently large to satisfy the above inequality, a state of incomplete indentation will result as illustrated in Fig.2. In this case the length of the contact region will depend on \( q_0 \) and is determined from the condition that the stress is bounded at the point \( x = a \) where the punch and the half-plane meet smoothly. For a bounded stress at \( x = a \), from (4.12)

\[ a = \frac{d + \bar{d}}{4\pi \varepsilon \delta} q_0, \]  

(4.15)
and hence

\[ P(x) = \frac{2\varepsilon \sin \pi \delta}{d + \bar{a}} \left( \frac{a - x}{a + x} \right)^{\bar{a}}. \tag{4.16} \]
CHAPTER 5
CONCLUSIONS

By applying Stroh's formalism and the method of analytical continuation, a general solution for the problems of punch indentation into an anisotropic elastic half-plane is derived in this paper. The generality of the present solution is shown as follows. (1) The half-plane is a general anisotropic medium. (2) The number of rigid punches indenting into the surface is arbitrary. (3) The location of each punch on the surface is arbitrary. (4) The profile of each punch is arbitrary but must be continuous, and in the case of sliding friction, the solution must be checked to ensure that the contact pressure is greater than zero. (5) The punches may complete adhere to the half-plane, or sliding with or without friction. (6) The cases of normal, tangential and rotary indentation are all included. For the purpose of verification and illustration, some special cases are deduced from this general solution such as normal and rotary indentation by a flat-ended punch into anisotropic or isotropic half-plane. The results show that our solutions are simple, general and exact.
REFERENCES


Hwu, C., 1992, Thermoelastic Interface Crack Problems in Dissimilar Anisotropic


23


APPENDIX A

Consider the eigenproblem
\[(\tilde{M}^{-1} + e^{2\pi i \delta} \tilde{M}^{-1}) \lambda = 0,\]  
(A1)

where \(\tilde{M}\) is the impedance matrix defined as \(\tilde{M} = -i \tilde{A}^{-1} \tilde{S}\). With the identities shown by Ting (1988), we have
\[\tilde{M}^{-1} = i \tilde{A} \tilde{B}^{-1} = (\tilde{I} - i \tilde{S}) \tilde{L}^{-1},\]  
(A2)

where \(\tilde{L}, \tilde{S}\) are real matrices composed of elasticity constants. Moreover, it can be shown that \(\tilde{S} \tilde{L}^{-1}\) is antisymmetric and \(\tilde{L}^{-1}\) is symmetric and positive definite.

Substituting (A2) into (A1) and for a nontrivial solution of \(\lambda\), we obtain
\[\| (1 - e^{2\pi i \delta}) \tilde{S} \tilde{L}^{-1} + i (1 + e^{2\pi i \delta}) \tilde{L}^{-1} \| = 0.\]  
(A3)

Since \(\tilde{L}^{-1}\) is positive definite, the determinant is nonzero if we set \(\delta = 0\). Hence \((1 - e^{2\pi i \delta}) \neq 0\), and (A3) may be rewritten as
\[\| \tilde{S} \tilde{L}^{-1} + i \beta \tilde{L}^{-1} \| = 0,\]  
(A4)

where
\[\beta = \frac{1 + e^{2\pi i \delta}}{1 - e^{2\pi i \delta}}.\]

Rearranging the above relation, we have
\[e^{2\pi i \delta} = -\frac{1 - \beta}{1 + \beta}.\]

It can be shown that if \(\delta\) is a root, so is \(\delta + n\) where \(n\) is an integer. If only \(-1 < \text{Re}(\delta) \leq 0\) are considered (Ting, 1986), we have
\[\delta = -\frac{1}{2} + \frac{i}{2\pi} \ln \frac{1 + \beta}{1 - \beta}.\]
The theorem proved by Ting (1986) states that:

Let $\beta$ be a root of the $3 \times 3$ determinant

$$
\| W + i\beta D \| = 0,
$$

where $\tilde{D}$ is a real, symmetric and positive definite matrix, while $\tilde{W}$ is a real antisymmetric matrix, then

$$
-\frac{1}{2} \mathrm{tr}(\tilde{W} \tilde{D}^{-1})^2 > 0,
$$

and the three roots are all real given by

$$
\beta = 0, \quad \beta = \pm \left[ -\frac{1}{2} \mathrm{tr}(\tilde{W} \tilde{D}^{-1})^2 \right]^{\frac{1}{2}}.
$$

By the above theorem, and replacing $\tilde{W}, \tilde{D}$ by $S\tilde{L}^{-1}, \tilde{L}^{-1}$, we have the solution for the eigenproblem (A1). The results are

$$
\delta_\alpha = -\frac{1}{2} + i \epsilon_\alpha, \quad \alpha = 1, 2, 3, \quad (A5)
$$

where

$$
\begin{align*}
\epsilon_1 &= \epsilon = \frac{1}{2\pi} \ln \frac{1 + \beta}{1 - \beta}, \\
\epsilon_2 &= -\epsilon, \\
\epsilon_3 &= 0, \\
\beta &= \left[ -\frac{1}{2} \mathrm{tr}(S^2) \right]^{\frac{1}{2}}. \quad (A6)
\end{align*}
$$
APPENDIX B

Consider the integral

\[ \tilde{j}(z) = \int_L \frac{1}{t - z} \left[ \tilde{X}_0^+(t) \right]^{-1} g(t) dt, \quad (B1) \]

where \( L \) is the union of a finite number of arcs \( L_1, L_2, \cdots, L_n \) and \( \tilde{X}_0(z) \) is the Plemelj function satisfying the relation

\[ \tilde{X}_0^+(t) + \overline{M} M^{-1} \tilde{X}_0^-(t) = 0. \quad (B2) \]

Suppose that \( g(t) \) is a polynomial, a situation which often occurs in practice. Then the integral along each \( L_k \) may be expressed in terms of an integral along a lacet \( C_k \) surrounding \( L_k \) as shown in Fig.3, and assume that \( z \) remains outside these lacets.

The contour integral around the lacet \( C_k \) may be represented as

\[
\int_{C_k} \frac{1}{\zeta - z} \left[ \tilde{X}_0(\zeta) \right]^{-1} g(\zeta) d\zeta = \int_{L_k} \frac{1}{t - z} \left[ \tilde{X}_0^+(t) \right]^{-1} g(t) dt - \int_{L_k} \frac{1}{t - z} \left[ \tilde{X}_0^-(t) \right]^{-1} g(t) dt \\
+ \lim_{\rho \to 0} \int_{|\zeta - a_k| = \rho} \frac{1}{\zeta - z} \left[ \tilde{X}_0(\zeta) \right]^{-1} g(\zeta) d\zeta \\
+ \lim_{\rho \to 0} \int_{|\zeta - b_k| = \rho} \frac{1}{\zeta - z} \left[ \tilde{X}_0(\zeta) \right]^{-1} g(\zeta) d\zeta.
\]

It may be shown that the last two integrals above tend to zero as \( \rho \to 0 \). Hence from (B2)

\[
\int_{C_k} \frac{1}{\zeta - z} \left[ \tilde{X}_0(\zeta) \right]^{-1} g(\zeta) d\zeta = \int_{L_k} \frac{1}{t - z} \left[ \tilde{X}_0^+(t) \right]^{-1} (I + \overline{M} M^{-1}) g(t) dt,
\]

then the integral \( \tilde{j}(z) \) may be expressed in the form

\[ \tilde{j}(z) = \int_C \frac{1}{\zeta - z} \left[ \tilde{X}_0(\zeta) \right]^{-1} (I + \overline{M} M^{-1})^{-1} g(\zeta) d\zeta, \quad (B3) \]

where \( C \) is the union of the lacets \( C_1, C_2, \cdots, C_n \).

Replacing \( C \) by a counterclockwise circle contour \( C_\infty \) at a large distance \( R \), we can obtain that

\[ \tilde{j}(z) = 2\pi i S - \int_{C_\infty} \frac{1}{\zeta - z} \left[ \tilde{X}_0(\zeta) \right]^{-1} (I + \overline{M} M^{-1})^{-1} g(\zeta) d\zeta, \quad (B4) \]
where \( S \) is the sum of residues of the poles of the integrand in (B3) lying between \( C \) and \( C_\infty \). The second term has the form

\[
\lim_{R \to \infty} \int_0^{2\pi} \frac{R e^{i\theta}}{R e^{i\theta} - z} \left[ X_0(Re^{i\phi}) \right]^{-1} (\bar{\zeta} + \bar{M}M^{-1})^{-1} g(Re^{i\phi}) \, i \, d\theta, \tag{B5}
\]

where \( R \) is the radius of the contour \( C_\infty \). It can be shown that only terms independent of \( Re^{i\theta} \) can contribute to the above integral. Then with a given function \( g(t) \), the integral \( j(z) \) can be explicitly evaluated from (B4) and (B5).

For example, consider an integral along a single line \( L = (-a, a) \) and let \( g(t) = \bar{g} \), where \( \bar{g} \) is a given constant vector. The sum of the residues is

\[
S = \left[ X_0(z) \right]^{-1} (\bar{\zeta} + \bar{M}M^{-1})^{-1} \bar{g}. \tag{B6}
\]

To calculate the integral shown in (B5), by (3.3d) we express \( \Gamma_\alpha^{-1}(\zeta) \) for large \( |\zeta| \) as

\[
\Gamma_\alpha^{-1}(\zeta) = (\zeta + a)^{1+\delta_\alpha} (\zeta - a)^{-\delta_\alpha}
= \zeta + 2ia\epsilon_\alpha + O\left(\frac{1}{\zeta}\right), \quad \alpha = 1, 2, 3.
\]

Hence (B5) becomes

\[
\lim_{R \to \infty} \int_0^{2\pi} \left( 1 + \frac{z}{Re^{i\theta}} + \cdots \right) \ll Re^{i\theta} + 2ia\epsilon_\alpha + \cdots \gg i \, d\theta \, A^{-1}(\bar{\zeta} + \bar{M}M^{-1})^{-1} \bar{g} \tag{B7}
\]

\[
= 2\pi i \ll z + 2ia\epsilon_\alpha \gg A^{-1}(\bar{\zeta} + \bar{M}M^{-1})^{-1} \bar{g}.
\]

From (B4), (B6) and (B7) we obtain the final result of \( j(z) \) as

\[
\dot{j}(z) = 2\pi i \left\{ \left[ X_0(z) \right]^{-1} \ll z + 2ia\epsilon_\alpha \gg A^{-1} \right\} (\bar{\zeta} + \bar{M}M^{-1})^{-1} \bar{g}. \tag{B8}
\]
Fig. 1 Notation of the half-plane
Fig. 2 A wedge-shaped punch under normal pressure
\( \cdot \mathbb{Z} \)

Fig. 3 A lacet integral contour