Anisotropic Elastic Inclusion Problems

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異向性彈性介質之相關問題

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一、中文摘要（關鍵詞：異向性、彈性介質、剛體介質、孔洞、任意荷重、差排）

針對二維異向性彈性體，以史拓公式結合保角映射和解析連續之方法，對於異向圓形介質之無限域彈性體，承受任意外力作用，可得廣義之解析解。同時，對於剛性介質或孔洞之特殊問題亦可解得。此廣義異向性彈性體之材質，並未限定任何材料之對稱性，亦即此二維推導，可涵蓋同平面和反平面問題，亦涵蓋同平面與反平面之耦合行之問題。此解剖形狀，可隨著其短軸之長度，縮減為零，即成線狀，伸長為長軸，則成一圓形。此無限域之應力和位移變形之解，可表為複數矩陣之形態。對沿介質邊界上之應力值，亦可求得經簡化之解。由於目前文獻中，並未出現此方面之廣義解，以驗證本文所推導之解是否正確，是故只能與一些特殊問題之現有解析解作比較，經此程序，可以證明本文之解法是正確且廣義。對於一些有趣問題，如在邊材中有集中荷重或差排出現，或者在無限遠處承受均佈力，都可求出其解析解，而其中集中荷重之解，可作為邊界元差法之格林函數，用來求解含孔洞或介質之有限域平面問題，或著是此格林函數已滿足殼面邊界條件，不必沿殼面邊界上加以離散化，只要在外圍有限域邊界上略加離散區間，即可獲得相當準確之值。

英文摘要 (Keywords: (Anisotropic, Elastic Inclusions, Rigid Inclusions, Holes and Cracks, Arbitrary Loading, Dislocation)
二、研究目的

研究单一層之弾性力學行為時，其幾何
和構造，可分別視為介質和基材。而從巨觀力
學觀點，單一層構可視為特殊正向性材科，複
合層構可為古典層板理論所用一類異向性物
體，故對異向性物質之分析，將其解的幾觀或
巨觀力學行為是相當重要的。本文主要著重在
研究具介質或孔洞之異向性彈性體之行為。

對二維異向性彈性體，利用史隆公式以求出
其解解析，是相當簡易且具靈活之方法，結合此
解析解和邊界元法更可分析含孔洞或介質之
有限域板之問題。半世紀來，有關含介質之弾性
體，應力場之決定，已被詳細探討，然研究探討
之範圍，彈性體雖然包括等向性材料、正向性材
料和異向性材料，但對異向性材料而言，行
為均侷限於作用於有限領域之均勻力。本文利用
史隆公式結合解析邊緣法(Muskheilishvili,1954)
類似Suo(1990)所提出之方法，可求出含圓形異
向性彈性體介質之無限域異向性基材承受任意大
力作用下之解析解，同時亦利用幾何概念，可
求出廣義解析解，求解過程相當簡潔流暢，所
得之應力或形變之解析解亦清楚明確。

三、研究方法及成果

對於二維的異向性彈性力學，文獻上有兩
種不同的方法：（一）Lekhnitskii的方法，（二）史
隆公式。兩種出發點不同，但兩者一解的表
示法極為相似。本文是採用史隆公式，史隆公式
的基本精神是設假三個方向的位移只跟x_1，x_2
有關，然後導出廣義解析並於推導過程中建立一
些與材料係數相關之恆等式。利用此一廣義解
配合恆等式，即可求出均勻域和流形域外的解，
可導出一有限域之基材含一開圓形介質，受任
一荷重下的廣義解析解。

對任一荷重所對應之復變函數f(s)本文
將它分兩類（一）以Taylor級數展開，（二）以
Laurent級數展開。

使用交換面連續的條件，若f(s)為分類
（一），可得弾性介質之廣義解析解為

\[ s_3 = f(s) + f(s) + \frac{1}{2} \left( \frac{d}{ds} f(s) + \frac{d}{ds} f(s) \right) \]

\[ s_2 = f(s) + f(s) + \frac{1}{2} \left( \frac{d}{ds} f(s) + \frac{d}{ds} f(s) \right) \]

\[ \text{其中} \]

\[ f(s) = B_2^{-1} \sum_{k=1}^{\infty} (B_2 f_2(s_2) + B_2 f_2(s_2)) s_2^{k-1} - B_2^{-1} B_2 f_2(1/s) \]

\[ f(s) = \sum_{k=1}^{\infty} f_2(s_2) s_2^{k-1} \]

\[ f(s) = \sum_{k=1}^{\infty} f_2(s_2) s_2^{k} \]

\[ f(s) = \sum_{k=1}^{\infty} f_2(s_2) s_2^{k} \]

符號說明請參照報告。

解此一解析解的正確性首先探討文獻
中所列荷重下的問題，結果與文獻所得完全相同，
則再對新的物理問題，如集中荷重，做詳加的
探討，發現弾性線性介質，不論有無和異向性
的問題，在鍵狀尖端的應力具有異常性。

另外本文也求得彈性介質和差排之間的交
互作用，並以一系列之映射力等線上界線，探討
非等向性之影響。

至於一有限域物體含有一開圓形孔洞承受任
意荷重下的問題，使用自由邊界孔洞之條件和
可解析連續方法可得廣義解為

\[ s = A \left( f(s) - f(s) \right) + \frac{1}{2} \left( \frac{d}{ds} f(s) + \frac{d}{ds} f(s) \right) \]

\[ s = B \left( f(s) - f(s) \right) + \frac{1}{2} \left( \frac{d}{ds} f(s) + \frac{d}{ds} f(s) \right) \]

其中

\[ f(s) = -B_2^{-1} B_2 f_2(1/s) \]

根據此一廣義解，本文解決了以下三個較
受重視的問題：（一）集中荷重，（二）無窮遠
處之均勻力，（三）差排和洞的交互作用。類
似孔洞問題利用連続解析方法，可得廣義解如
式(2)，但f(s)改變為

\[ f(s) = -A_2^{-1} A_2 f_2(1/s) + \frac{1}{2} \left( \frac{d}{ds} f_2(s) + \frac{d}{ds} f_2(s) \right) \]
其中ω代表剛體介質相對於基材的旋轉量，之量是可由作用於剛性介質表面力所引起的力矩和爲零之條件求得，同剛性介質與孔洞的問題，根據此一理論可解決下列問題：（一）等荷重，（二）均布荷重，（三）垂排和剛性介質的交互作用。

二、結論與討論

以史密斯公式為基礎結合正交映射和解析近似方法解決含異型異向性彈性介質之無限域內異向性基材，受任一荷重之問題，可得一複變數型態之廣義解，此解包括彈性介質問題，及孔洞問題和剛性介質問題之解析解。圖一所示彈性介質承受均布荷重之問題而應力，其結果與文獻所示相符。圖二則為彈性介質與垂排交互影響之映像力等高線圖。圖三、四分別為孔洞與剛性介質之結果。

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摘要

針對二維異向性彈性體，以史其公式結合保角映射和解析方法，對於具有模形介質的無限域彈性體，承受任意外力作用，可得廣義之解析解。同時，對於剛性介質或孔洞之特殊問題亦可解得。此廣義異向性彈性體之材質，並未限定任何材料之對稱性，亦即此二維推導，可涵蓋同平面和反平面問題，亦涵蓋同平面與反平面之耦合行行為之問題。此橢圓形狀，可隨著調整其短軸之長度，縮減為零，即成線狀，伸長為長軸，則成一圓形。此無限域之應力和位移變形之解，可表為複數矩陣之形態。對沿介質邊界上之應力值，亦可求得經簡化之解。由於截至目前文獻中，並未出現此方面之廣義解，以驗證本文所推導之解是否正確，是故只能與一些特殊問題之現有解析解作比較，經此程序，可以證明本文之解法是正確且廣義。對於一些有趣問題，如在基材中有集中荷重或差排出現或者在無限遠處承受均布力，都可求出其解析解，而其中集中荷重之解，可作爲邊界元素法之格林函數，用來求解含孔洞或介質之有限域平面問題，優點是此格林函數已滿足橢圓邊界條件，不必沿橢圓邊界上加以離散化，只要在外圍有限域邊界上略加離散區間，即可獲得相當精確之值。

關鍵詞：異向性，彈性介質，剛性介質，孔洞，任意荷重，差排。
ABSTRACT

By combining the method of Stroh's formalism, the concept of perturbation, the technique of conformal mapping and the method of analytical continuation, a general analytical solution for the elliptical anisotropic elastic inclusions embedded in an infinite anisotropic matrix subjected to an arbitrary loading has been obtained in this paper. The inclusion as well as the matrix are of general anisotropic elastic materials which do not imply any material symmetry. The special cases when the inclusion is rigid or a hole are also studied. The arbitrary loadings include inplane and antiplane loadings. The shapes of ellipses cover the lines or circles when the minor axis is taken to be zero or equal to the major axis. The solutions of the stresses and deformations in the entire domain are expressed in complex matrix notation. Simplified results are provided for the interfacial stresses along the inclusion boundary. Some interesting examples are solved explicitly such as point forces or dislocations in the matrix, and uniform loadings at infinity. Since the general solutions have not been found in the literature, comparison is made with some special cases of which the analytical solutions exist, which shows that our results are exact and universal.

Keywords: Anisotropic, Elastic Inclusions, Rigid Inclusions, Holes and Cracks, Arbitrary Loading, Dislocation
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CHAPTER I

INTRODUCTION

Determination of stress field for an elastic body in the presence of inclusions has been extensively studied for almost half a century. An extent review on the subject can be found in Mura’s book (1982) and his recent review article (Mura, 1988). A great development on the inclusion problems was done by Eshelby (1957,1959). He introduced a point force concept to solve three-dimensional inclusion problems, which provides a high degree of flexibility and generality. However, it involves a set of integrals which in general are analytically intractable. Based upon the Eshelby’s concept, many researchers investigated the two-dimensional elliptic inclusion problems and obtained analytic solution. Jaswon and Bhargave (1961) considered the isotropic case under the condition of plane strain (or plane stress). Their analysis was extended by Wills (1964) to the materials with cubic symmetry. Subsequent anisotropic analysis was performed by Bhargave and Radhakrishna (1966) for an anisotropic medium with two symmetry planes (orthotropic) and by Chen (1967) for monoclinic materials. Yang and Chou (1976) employed the method developed by Eshebly et al. (1953) to solve for general anisotropic material. The inplane and antiplane deformation may be coupled. They also considered the special case under antiplane strain (Yang and Chou, 1977). To obtain the real form solutions of interface for elliptic inclusion, Hwu and Ting (1989) used sum rules and identities (Ting, 1988a, 1988b; Ting and Hwu, 1988; Hwu and Ting, 1990). All the above studies were focused on the problems under uniform loading. The cases of general
loading were considered only for isotropic materials by Aderogba (1973) for circular shape and by Stagni (1982) for elliptic shape. Both of them introduced perturbation concept. The advantage of perturbation concept is that by means of solutions for a perfect infinite region, one can easily extend it to the solutions for an infinite region with an inclusion.

In this study, the Stroh's formalism for anisotropic elasticity combined with the method of analytical continuation (Muskhelishvili, 1954), which is similar to the one proposed by Suo (1990), is developed to solve the general analytical solution for the elliptical anisotropic elastic inclusions imbedded in an infinite anisotropic matrix subjected to an arbitrary loading. Moreover, the concept of perturbation is applied to formulate the general solutions. A transformation function which maps the ellipse onto a unit circle is introduced. However, a discontinuity problem occurs when the transformation is required to be single-valued and conformal in the entire domain including the matrix and inclusion. This is remedied by the way similar to those proposed by Stagni (1982) for isotropic materials, i.e., a restricted condition is introduced to force the continuity. The general loading conditions considered include the cases of point singularities such as point forces or dislocations. The analytical closed form solutions presented in this paper are universal in the sense of materials (anisotropic elastic), loadings (arbitrary) and geometries (elliptic). The solutions of stresses and deformations in the entire domain are expressed in complex matrix notation. Simplified results are provided for the interfacial stresses along the inclusion boundary through the use of identities developed in the literature.

The cases of an elliptic rigid inclusion or hole are also considered in this study.
It is found that by use of analytic continuation very simple procedures are involved and solutions with a surprisingly simple form are obtained. The case of anisotropic medium with an elliptic hole under uniform loading or pure bending can be found in Savin (1961) and Lekhnitskii (1968). Using Lekhnitskii's approach, the analytic solutions of infinite media with an elliptic hole subjected to a point force were obtained in series form by Tarn and Chen (1987). In addition, Kamel and Liaw (1989a,b) found the solution for the cases of point forces and concentrated moments by employing the superposition method and Cauchy's integral theorem (Muskhelishvili, 1954). In contrast to the above two approaches, the method developed in this study is much simpler and the solution is general with regard to loading and material properties. As for the cases of rigid inclusions, Yang and Chou (1978) obtained the solutions for orthotropic materials. Santare and Keer (1986) treated the problem of interaction between an edge dislocation and a rigid elliptical inclusion for isotropic materials.

To show the generality of the present results. Some special and interesting examples are solved explicitly. The cases of point singularity, which are important for practical application, is of significant concern. It is well known that the solution of point forces can be employed as the fundamental solution for the boundary element method (Brebbia, et al., 1984). Moreover, the solutions of dislocations are frequently used as kernel function of integral equation to consider the interactions between inclusions and cracks (Erdogan, et al., 1974; Pattan and Santare, 1990). The problem of a circular elastic inhomogeneity near an edge dislocation was solved in terms of Airy's stress potentials by Dundurs and Mura (1964). Later, Dundurs
and Sendeckyj (1965) solved the related problem where the dislocation is within
the circular inhomogeneity. Recently, the analytic solutions of isotropic elliptic in-
clusion were obtained by Stagni and Lizzio (1983) for a dislocation located outside
an elliptic inhomogeneity and by Warren (1983) for a dislocation inside an elastic
elliptic inhomogeneity.

"Elliptic" includes the special case such as crack or line inclusion when the mi-
nor axis of ellipse tends to zero. Crack problems has been widely studied for a long
time. For a rigid line inclusion in an isotropic elastic matrix subjected to uniform
loading, the analytic solutions were obtained by Wang et al. for plane problem
(1985) and antiplane problem (1986) where the methods of Eshelby (1957,1959)
and Mushhelishvili (1954) were employed. The same plane problem also has been
considered by Ballarini (1987) using the method of integral transform. As for the
problem of elastic line inclusion, Erdogan and Gupta (1972) investigated problem
of bonded isotropic material containing a flat inclusion. Furthermore, Li and Ting
(1989) considered a line inclusion in anisotropic elastic solids. They neglected bend-
ing rigidity of elastic line inclusion and concluded that the nature of stress field at
the tip of line inclusion exhibits a square root singularity, which is similar to those
in cracks or rigid line inclusions. In this report, the elastic line inclusion is simu-
lated by letting $b$ approach to zero. From the results for the hoop stress $\sigma_{\theta\theta}$ along
$x - 1$ versus the ratio of the minor axis to the major axis ($b/a$), we observe that
a constant value of $\sigma_{\theta\theta}$ is approached when $b/a \to 0$ for a variety of the hardness
index. This constant depends on the index. Hence, the singular behavior does not
exist in elastic line inclusion which is different from the cases of cracks or rigid line

* 4
inclusions.

This study is divided into six chapters. The first chapter describes the problems, the objects of this study and reviews of literature concerning the emphasized subject. The theory of Stroh’s formalism including some useful identities is introduced in Chapter II. Based upon the Stroh’s formalism, the general solutions for infinite anisotropic media with an elliptic inclusion under arbitrary loading are given in Chapter III. Chapter III is also devoted to develop some important methods such as analytical continuation, conformal mapping and perturbation concept. Some interesting examples are provided in Chapter III. The cases for uniform loading have appeared in the literature are considered for the purpose of verification of the present method. The cases for elastic bodies under point forces and dislocations are very important for practical application. The problems of holes and cracks are considered in Chapter IV. Infinite or finite media with rigid inclusions are considered in Chapter V. Chapter VI makes some conclusions.
CHAPTER II

TWO-DIMENSIONAL ANISOTROPIC ELASTICITY

2.1 General Solutions for Two-Dimensional Anisotropic Elasticity

For two-dimensional anisotropic elasticity, there are two different formulations in the literature. One is the Lekhnitskii’s approach (1968) which starts with the equilibrated stress functions then compatibility equations, the other is Stroh’s formalism (1958) which starts with the compatible displacements then equilibrium equations. The equivalency of these two formulations has been discussed in (Suo, 1990). In this paper, we follow Stroh’s formalism due to its elegance and simplicity and the notation in (Hwu and Ting, 1989) is employed. In a fixed rectangular coordinate system \( \mathbf{x}_i, i = 1, 2, 3 \), let \( u_i, \sigma_{ij}, \epsilon_{ij} \) be, respectively, the displacement, stress and strain. With body forces neglected, the strain-displacement equations, the stress-strain laws, and the equations of equilibrium are

\[
\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) , \tag{2.1}
\]

\[
\sigma_{ij} = C_{ijks} \epsilon_{ks} , \tag{2.2}
\]

\[
\sigma_{ij,j} = C_{ijk\ell} u_{k,\ell j} = 0 , \tag{2.3}
\]

where repeated indices imply summation, a comma stands for differentiation and \( C_{ijks} \) are the elastic constants which are assumed to be fully symmetric and positive definite. If we consider a two-dimensional deformation in which \( u_i, i = 1, 2, 3, \)
depend on $x_1$ and $x_2$ only, the displacement $u_i$ can be written as

$$u_i = a_i f(z), \quad z = x_1 + px_2.$$  \hfill (2.4)

By substituting (2.4) into (2.3), the general solution for $u_i$ can be written in matrix notation as

$$u_\alpha = \sum_{\alpha=1}^6 a_\alpha f_\alpha(z_\alpha), \quad z_\alpha = x_1 + p_\alpha x_2, \quad (\alpha \text{ not summed}),$$  \hfill (2.5)

in which $f_1, f_2, \cdots$ are arbitrary functions of their arguments and $p_\alpha$ and $a_\alpha$ are the eigenvalues and eigenvectors of the following eigenrelation:

$$\{Q + p(R + R^T) + p^2T\}a = 0.$$  \hfill (2.6)

In (2.6) the superscript $T$ stands for the transpose and $Q, R, T$ are the $3 \times 3$ real matrices given by

$$Q_{ik} = C_{i1k1}, \quad R_{ik} = C_{i1k2}, \quad T_{ik} = C_{i2k2}.$$  \hfill (2.7)

We see that $Q$ and $T$ are symmetric and positive definite if the strain energy is positive (Esbelby, Read and Shochley, 1953). Because the eigenvalue $p$ of (2.6) cannot be real if the strain energy is positive, we can obtain three pairs of complex conjugates for $p$. Let

$$p_{\alpha+3} = \bar{p}_\alpha, \quad \text{Im}(p_\alpha) > 0, \quad \alpha = 1, 2, 3,$$

where an overbar denotes the complex conjugate and $\text{Im}$ stands for the imaginary part. We then have

$$a_{\alpha+3} = \bar{a}_\alpha, \quad \alpha = 1, 2, 3.$$
For the displacement \( \mathbf{u} \) to be real, we let

\[ f_{\alpha+3} = \mathbf{f}_{\alpha}, \quad \alpha = 1, 2, 3, \]

and (2.5) becomes

\[ \mathbf{u} = 2 \text{Re} \left\{ \sum_{\alpha=1}^{3} \mathbf{a}_\alpha f_\alpha(z_\alpha) \right\}, \tag{2.8} \]

in which \( \text{Re} \) stands for the real part.

Introducing the vector

\[ \mathbf{b} = (R^T + pT)\mathbf{a} = -\frac{1}{p}(Q + pR)\mathbf{a}, \tag{2.9} \]

where the second equality comes from (2.6), the stresses \( \sigma_{ij} \) obtained by substituting (2.5) into (2.1) and (2.2) can be written as

\[ \sigma_{i1} = -\phi_{i,2}, \quad \sigma_{i2} = \phi_{i,1}, \tag{2.10} \]

where \( \phi \) is the stress function

\[ \phi = 2 \text{Re} \left\{ \sum_{\alpha=1}^{3} \mathbf{b}_\alpha f_\alpha(z_\alpha) \right\}, \tag{2.11} \]

and \( \mathbf{b}_\alpha \) is related to \( \mathbf{a}_\alpha \) through (2.9). More generally, if \( \mathbf{t} \) is the surface traction at a point on a curve boundary, then

\[ \mathbf{t} = \partial \phi / \partial s, \tag{2.12} \]

where \( s \) is the arc length measured along the curved boundary in the direction such that, when one faces the direction of increasing \( s \), the material is located on the right-hand side. We see that (2.10) are special cases of (2.12) when the boundary is a plane parallel to the \( x_2 \)-axis or the \( x_1 \)-axis.
With similar reason as Suo (1990) that whether a function is analytic is not affected by different arguments \( z_\alpha = x_1 + p_\alpha x_2 \), \( \alpha = 1, 2, 3 \), for (2.8) and (2.11) another solution form appropriate for the method of analytic continuation is written as

\[
\begin{align*}
\underline{u} &= \underline{A} \underline{f}(z) + \underline{\bar{A}} \bar{\underline{f}}(z), \\
\underline{\phi} &= \underline{B} \underline{f}(z) + \underline{\bar{B}} \bar{\underline{f}}(z),
\end{align*}
\]

(2.13a)

where

\[
\underline{A} = \begin{bmatrix}
 a_1 & a_2 & a_3
\end{bmatrix},
\]

\[
\underline{B} = \begin{bmatrix}
 b_1 & b_2 & b_3
\end{bmatrix},
\]

(2.13b)

\[
\underline{f}(z) = \begin{bmatrix}
 f_1(z) & f_2(z) & f_3(z)
\end{bmatrix}^T.
\]

Note that the argument of each component function of \( \underline{f}(z) \) is written as \( z = x_1 + px_2 \) without referring to the associated eigenvalues \( p_\alpha \). Once the solution of \( \underline{f}(z) \) is obtained for a given boundary value problem, a replacement of \( z_1, z_2 \) or \( z_3 \) should be made for each component function to calculate field quantities from (2.8) and (2.11).

In many applications, \( f_1, f_2, f_3 \) have the same function form

\[
f_\alpha(z_\alpha) = q_\alpha f(z_\alpha), \quad \alpha \text{ not summed},
\]

where \( q_\alpha, \alpha = 1, 2, 3 \), are arbitrary complex constants. Equations (2.8) and (2.11) can then be written as

\[
\begin{align*}
\underline{u} &= 2 \text{Re}\{ \underline{A} \ll f(z_\alpha) \gg \underline{q} \}, \\
\underline{\phi} &= 2 \text{Re}\{ \underline{B} \ll f(z_\alpha) \gg \underline{q} \},
\end{align*}
\]

(2.14)

in which \( \underline{q} \) is the \( 3 \times 1 \) matrix whose elements are \( q_\alpha, \alpha = 1, 2, 3 \) and the angular bracket stands for the diagonal matrix, i.e., \( \ll f_\alpha \gg = \text{diag}\{ f_1 \ f_2 \ f_3 \} \), which will be used throughout this report.
2.2 The Sextic Formalism of Stroh

The two equations in (2.9) can be written in a standard eigenrelation

\[ \begin{align*}
\mathcal{N} \xi &= p \xi, \\
\mathcal{N} &= \begin{bmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ \mathcal{N}_3 & \mathcal{N}_1^T \end{bmatrix}, \\
\xi &= \begin{bmatrix} a \\ b \end{bmatrix}
\end{align*} \tag{2.15} \]

\[ \begin{align*}
\mathcal{N}_1 &= -\mathcal{T}^{-1} R^T, \\
\mathcal{N}_2 &= \mathcal{T}^{-1}, \\
\mathcal{N}_3 &= R_T^{-1} R^T - Q,
\end{align*} \tag{2.16} \]

where \( \mathcal{N}_2 \) and \( \mathcal{N}_3 \) are symmetry and \( \mathcal{N}_2 \) is positive definite. With the definition in (2.13b), the matrices \( \mathcal{A} \) and \( \mathcal{B} \) satisfy the following orthogonality relations as (Stroh, 1958; Ting, 1986)

\[ \begin{align*}
\mathcal{A}^T \mathcal{B} + \mathcal{B}^T \mathcal{A} &= \mathcal{I} = \mathcal{A}^T \mathcal{B} + \mathcal{B}^T \mathcal{A}, \\
\mathcal{A}^T \mathcal{B} + \mathcal{B}^T \mathcal{A} &= 0 = \mathcal{B}^T \mathcal{A} + \mathcal{A}^T \mathcal{B}, \\
\mathcal{A} \mathcal{A}^T + \mathcal{A} \mathcal{A}^T &= 0 = \mathcal{B} \mathcal{B}^T + \mathcal{B} \mathcal{B}^T, \\
\mathcal{B} \mathcal{A}^T + \mathcal{B} \mathcal{A}^T &= \mathcal{I} = \mathcal{A} \mathcal{B}^T + \mathcal{A} \mathcal{B}^T,
\end{align*} \tag{2.18} \]

where \( \mathcal{I} \) is the unit matrix. It follows from (2.18c,d) that three real matrices \( \mathcal{H}, \mathcal{S}, \) and \( \mathcal{L} \) can be defined as (Ting, 1986)

\[ \begin{align*}
\mathcal{H} &= 2i \mathcal{A} \mathcal{A}^T, \\
\mathcal{L} &= -2i \mathcal{B} \mathcal{B}^T, \\
\mathcal{S} &= -i(2 \mathcal{A} \mathcal{B}^T - \mathcal{I}).
\end{align*} \tag{2.19} \]
It can be shown that \( \EuScript{H} \) and \( \EuScript{L} \) are symmetric and positive definite (Chadwick and Smith, 1977).

2.3 Identities

To generalize the eigenrelation (2.15) one consider another formalism by assuming that

\[
\xi = a \hat{f}(z \cdot n + p(\theta)z \cdot m),
\]

where the symbol "\( \cdot \)" denote the inner product of two vectors and

\[
\psi^T(\theta) = (\cos \theta, \sin \theta, 0)
\]

\[
m^T(\theta) = (\sin \theta, \cos \theta, 0)
\]

and \( \theta \) is an arbitrary real parameter. With (2.20), equation (2.7) is replaced by

\[
Q_{ik} = C_{ijk} n_j n_k,
\]

\[
R_{ik} = C_{ijk} n_j m_k,
\]

\[
T_{ik} = C_{ijk} m_j m_k,
\]

and (2.15)-(2.17) become

\[
\mathcal{N}(\theta) \xi = p(\theta) \xi,
\]

\[
\mathcal{N}(\theta) = \begin{bmatrix}
\mathcal{N}_1(\theta) & \mathcal{N}_2(\theta) \\
\mathcal{N}_3(\theta) & \mathcal{N}_3^T(\theta)
\end{bmatrix}, \quad \xi = \begin{bmatrix}
a \\
b
\end{bmatrix}
\]

\[
\mathcal{N}_1(\theta) = -\mathcal{L}^{-1}(\theta) R^T(\theta),
\]

\[
\mathcal{N}_2(\theta) = \mathcal{T}^{-1}(\theta),
\]

\[
\mathcal{N}_3(\theta) = R(\theta) \mathcal{T}^{-1}(\theta) R^T - Q(\theta),
\]

As before, there are six eigenvalues \( p_\alpha(\theta), \alpha = 1, 2, \cdots, 6 \) which come in three pairs of complex conjugate. It can be shown (Chadwick and Smith, 1977) that when \( \mathcal{N}(\theta) \)
is simple or semi-simple $\xi$ is independent of $\theta$. It should be noted that (2.23) to (2.25) can reduce to (2.15) to (2.17) when $\theta = 0$. The eigenvalue $p_\alpha(\theta)$ are related to $p_\alpha$ by (Ting, 1982)

$$p_\alpha(\theta) = \frac{p_\alpha \cos \theta - \sin \theta}{p_\alpha \sin \theta + \cos \theta}. \quad (2.26)$$

Equation (2.23) written for $p(\theta) = p_1(\theta), p_2(\theta), p_3(\theta)$ can be combined into one compact form as

$$\begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \ll p_\alpha(\theta) \gg \\ \mathcal{B} & \ll p_\alpha(\theta) \gg \end{bmatrix}, \quad (2.27)$$

where $\mathcal{A}$ and $\mathcal{B}$ are defined in (2.13b). Postmultiplying both sides of (2.27) by $\mathcal{B}^T, \mathcal{A}^T$ and using (2.19), we can obtain

$$2 \begin{bmatrix} \mathcal{A} & \ll p_\alpha(\theta) \gg \mathcal{B}^T \\ \mathcal{B} & \ll p_\alpha(\theta) \gg \mathcal{B}^T \end{bmatrix} = N(\theta) \begin{bmatrix} \mathcal{I} - i\mathcal{S} & -i\mathcal{H} \\ i\mathcal{L} & \mathcal{I} - i\mathcal{S}^T \end{bmatrix}. \quad (2.28)$$

Substituting (2.24) into (2.28), we have the following identities:

$$2 \mathcal{A} \ll p_\alpha(\theta) \gg \mathcal{B}^T = N_1(\theta) - i[N_1(\theta) \mathcal{S} - N_2(\theta) \mathcal{L}],$$

$$= N_1(\theta) - i[\mathcal{S} N_1(\theta) + \mathcal{H} N_3(\theta)], \quad (2.29)$$

$$2 \mathcal{A} \ll p_\alpha(\theta) \gg \mathcal{A}^T = N_2(\theta) + i[N_1(\theta) \mathcal{H} - N_2(\theta) \mathcal{S}^T],$$

$$2 \mathcal{B} \ll p_\alpha(\theta) \gg \mathcal{B}^T = N_3(\theta) - i[N_3(\theta) \mathcal{S} - N_1^T(\theta) \mathcal{L}].$$

If we postmultiply both sides of (2.27) by $\mathcal{B}^{-1}, \mathcal{A}^{-1}$ and using (Ting, 1986)

$$\mathcal{A} \mathcal{B}^{-1} = -(\mathcal{S} + i\mathcal{I}) \mathcal{L}^{-1} = \mathcal{L}^{-1} (\mathcal{S}^T - i\mathcal{I}),$$

$$\mathcal{B} \mathcal{A}^{-1} = -(\mathcal{S}^T + i\mathcal{I}) \mathcal{H}^{-1} = -\mathcal{H}^{-1} (\mathcal{S} - i\mathcal{I}). \quad (2.30)$$

we can obtain the following identities

$$\mathcal{A} \ll p_\alpha(\theta) \gg \mathcal{A}^{-1} = (N_1(\theta) - N_2(\theta) \mathcal{H}^{-1} \mathcal{S}) + iN_2(\theta) \mathcal{H}^{-1},$$

$$\mathcal{A} \ll p_\alpha(\theta) \gg \mathcal{B}^{-1} = (N_3(\theta) + N_1(\theta) \mathcal{L}^{-1} \mathcal{S}^T) - iN_1(\theta) \mathcal{L}^{-1},$$

$$\mathcal{B} \ll p_\alpha(\theta) \gg \mathcal{B}^{-1} = (N_3^T(\theta) - N_2(\theta) \mathcal{S} \mathcal{L}^{-1}) - iN_3(\theta) \mathcal{L}^{-1},$$

$$\mathcal{B} \ll p_\alpha(\theta) \gg \mathcal{A}^{-1} = (N_3(\theta) - N_1^T(\theta) \mathcal{H}^{-1} \mathcal{S}) - iN_1^T(\theta) \mathcal{H}^{-1}. \quad (2.31)$$
The identities given in (2.30) and (2.31) will be useful in obtaining a real form solution to two-dimensional anisotropic elasticity problems.

Barnett and Lothe (1973) proposed an alternative expression for $\tilde{S}, \tilde{H}, \tilde{L}$ defined in (2.19) as

$$\tilde{S} = \frac{1}{\pi} \int_{0}^{\pi} \tilde{N}_1(\theta) \, d\theta,$$

$$\tilde{H} = \frac{1}{\pi} \int_{0}^{\pi} \tilde{N}_2(\theta) \, d\theta,$$

$$\tilde{L} = -\frac{1}{\pi} \int_{0}^{\pi} \tilde{N}_3(\theta) \, d\theta.$$  \hspace{1cm} (2.32)

In this way, the need to determine the eigenvalues $p_\alpha$ and eigenvectors $\tilde{A}$ and $\tilde{B}$ is circumvented and hence (2.32) is valid regardless of $\tilde{N}$ is simple, semisimple or non-semisimple.
CHAPTER III

ELASTIC INCLUSIONS

3.1 Conformal Mapping

Consider an elliptic anisotropic elastic inclusion embedded in an infinite anisotropic matrix (Figure 3.1). The contour of the elliptic boundary is represented by

\[ x_1 = a \cos \psi, \quad x_2 = b \sin \psi, \quad (3.1) \]

where \( \psi \) is a real parameter and \( a, \ b \) are lengths of semi-axes of ellipse. It is known that the transformation function,

\[ z_\alpha = \frac{1}{2} \{(a - ibp_\alpha)\zeta_\alpha + (a + ibp_\alpha)\frac{1}{\zeta_\alpha}\}, \quad (3.2a) \]

or

\[ \zeta_\alpha = \frac{z_\alpha + \sqrt{z_\alpha^2 - a^2 - p_\alpha^2 b^2}}{a - ip_\alpha b}, \quad (3.2b) \]

will map the points \( z_\alpha \) along the elliptic boundary a unit circle in \( \zeta_\alpha \)-domain.

The roots of the equation, \( \frac{dz_\alpha}{d\zeta_\alpha} = 0 \), are at

\[ \zeta_\alpha^* = \pm \sqrt{\frac{a + ibp_\alpha}{a - ibp_\alpha}} = \pm \sqrt{m_\alpha} e^{i\theta_\alpha}, \quad (3.3) \]

where \( \sqrt{m_\alpha} \) and \( \theta_\alpha \) denote, respectively, the modulus and argument of the critical points \( \zeta_\alpha^* \). Since \( \sqrt{m_\alpha} < 1 \), which can easily be proved if the imaginary part of \( p_\alpha \) has
been set to be positive, the transformation is single-valued and conformal outside the elliptic inclusion. However, the inside region is double-valued and nonconformal. Figure 3.2 shows the transformation among \( z \)-plane, \( z_\alpha \)-plane and \( \zeta_\alpha \)-plane. It can be seen that there are two different \( \zeta_\alpha \) inside the unit circle corresponding to one \( z_\alpha \) inside the elliptic inclusion. To have a one-to-one transformation, we designate the point nearest the unit circle to be the mapped point. For this choice, a discontinuity problem may happen when two originally continuous points \((z_\alpha)_1\) and \((z_\alpha)_2\) are mapped onto \((\zeta_\alpha)_1\) and \((\zeta_\alpha)_2\) shown in Figure 3.2. Actually the points \((\zeta_\alpha)_1 = \sqrt{m_\alpha} \sigma\) and \((\zeta_\alpha)_2 = \sqrt{m_\alpha} e^{2i\theta_\alpha} / \sigma\) correspond to the same point in the \( z_\alpha \)-plane, where \( \sigma = e^{i\psi} \) denotes the points located on the unit circle. Hence, the transformation function (3.2) now maps the whole \( z_\alpha \)-plane, cut along a slit, into the \( \zeta_\alpha \)-plane deprived of the circle of radius \( \sqrt{m_\alpha} \). To remedy this discontinuity, i.e., eliminating the slit which does not exist in our problem, the following restriction should be satisfied

\[
f(\sqrt{m_\alpha} \sigma) = f(\sqrt{m_\alpha} e^{2i\theta_\alpha} / \sigma). \tag{3.4}
\]

By applying the conformal mapping technique described above and the perturbation concept given in (Stagni, 1982), the general solutions for the inclusion problems can now be written in terms of the variables \( \zeta_\alpha \), i.e.

\[
\begin{align*}
\psi_1 &= A_1 \left[ f_0(\zeta) + f_1(\zeta) \right] + \bar{A}_1 \left[ \bar{f}_0(\zeta) + \bar{f}_1(\zeta) \right], \quad \zeta \in S_1 \tag{3.5a} \\
\varphi_1 &= B_1 \left[ f_0(\zeta) + f_1(\zeta) \right] + \bar{B}_1 \left[ \bar{f}_0(\zeta) + \bar{f}_1(\zeta) \right]
\end{align*}
\]

and

\[
\begin{align*}
\psi_2 &= A_2 \bar{f}_2(\zeta^*) + \bar{A}_2 \bar{f}_2(\zeta^*) \\
\varphi_2 &= B_2 \bar{f}_2(\zeta^*) + \bar{B}_2 \bar{f}_2(\zeta^*) \quad \zeta^* \in S_2 \tag{3.5b}
\end{align*}
\]
where the subscripts 1 and 2 denote, respectively, the matrix and inclusion. \( \zeta_\alpha^* \) is the mapped point of \( z_\alpha^* = x_1 + p_\alpha^* x_2 \) where \( p_\alpha^* \) is the material eigenvalue of the inclusion. \( \mathcal{f}_\alpha \) represents the function associated with the unperturbed elastic field which is related to the solutions of homogeneous media and is holomorphic in the entire domain except some singular points such as the points under concentrated forces or dislocations, and the points at zero or infinity. \( \mathcal{f}_1 \) (or \( \mathcal{f}_2 \)) is the function corresponding to the perturbative field of matrix (or inclusion) and is holomorphic in region \( S_1 \) (or \( S_2 \)) except some singular points. \( S_1 \) and \( S_2 \) denotes, respectively, the regions occupied by the matrix and inclusion. Hence, in \( \zeta_\alpha \)-plane, \( S_1 \) is the region outside the unit circle, while \( S_2 \) is the region of the annual ring between the unit circle and the circle of radius \( \sqrt{m_\alpha} \). Since \( \mathcal{f}_2 \) is holomorphic in the annual ring, it can be represented by Laurent's expansion,

\[
\mathcal{f}_2(\zeta^*) = \sum_{k=-\infty}^{\infty} \zeta_k \zeta^* k. \tag{3.6a}
\]

Satisfaction of (3.4) gives

\[
\zeta_{-k} = \Gamma_k^* \zeta_k, \tag{3.6b}
\]

where

\[
\Gamma_k^* = \langle \left( \frac{a + i b p_\alpha^*}{a - i b p_\alpha^*} \right)^k \rangle. \tag{3.6c}
\]

Note that the general solutions of (2.13) and (3.5) require that each component of the column vector \( \mathcal{f} \) have different argument \( z_\alpha \) or \( \zeta_\alpha \). Hence, equation (3.6a) has the implicit meaning that

\[
\mathcal{f}_2(\zeta^*) = \sum_{k=-\infty}^{\infty} \{ (c_k)_1 \zeta_1^* k \quad (c_k)_2 \zeta_2^* k \quad (c_k)_3 \zeta_3^* k \}^T.
\]
where $(c_\alpha)_{\alpha = 1, 2, 3}$ are the components of $\zeta_k$.

3.2 General Loading Conditions

For a given loading condition, $f_0$ can be obtained immediately since it is related to the solutions of homogeneous media. However, it is not necessary to be exactly the same as the solutions of homogeneous media. The choices of $f_0$ depend on the convenience in calculation. The final solutions for the stresses and deformations in the entire domain will not be influenced by the choices of $f_0$. To have a better understanding about the choices, two special examples are discussed in the following.

(a) A dislocation $\hat{b}$ or point force $\hat{p}$ at $z_\alpha = \hat{z}_\alpha$

Consider a dislocation line in the direction perpendicular to $x_1 x_2$ plane with Burger vector $\hat{b}$, and a point force uniformly distributed along $x_3$-axis with force per unit length $\hat{p}$. Both singularities are at the point $(\hat{x}_1, \hat{x}_2)$. If $f_0$ is chosen to represent exactly the solutions of homogeneous media, it may be written as

$$f_0(\zeta) = \ll \log(z_\alpha - \hat{z}_\alpha) \gg \varrho ,$$

(3.7)

where $\varrho$ is to be determined in terms of $\hat{b}$ and $\hat{p}$. Using the condition that around any circle enclosing the point $(\hat{x}_1, \hat{x}_2)$, the total displacements and forces resulting from $f_0(\zeta)$ are $\hat{b}$ and $\hat{p}$, we can obtain

$$2Re\{iB\varrho\} = \hat{p}/2\pi ,$$

$$2Re\{iA\varrho\} = \hat{b}/2\pi .$$

(3.8)

To find $\varrho$ from (3.8), we use the orthogonality relations given in (2.18a,b) and obtain

$$\varrho = \frac{1}{2\pi i} (A^T \hat{p} + B^T \hat{b}) .$$

(3.9)
However, it is inconvenient in calculation when our general solution is expressed in terms of the variable $\zeta_\alpha$ not $z_\alpha$. An alternative choice for $f_\alpha$ is

$$f_\alpha(\zeta) = \ll \log(\zeta - \zeta_\alpha) \gg g,$$  \hspace{1cm} (3.10)

where $g$ is the same as (3.9). This expression is more convenient than the one given in (3.7). Moreover, it also reflects the singularity characteristics of the original problems.

(b) Uniform loading applied at infinity

The exact solution corresponding to the homogeneous media is (Ting, 1988a)

$$f_\alpha(\zeta) = \ll z_\alpha \gg g,$$

$$= \frac{1}{2} \ll a - ibp_\alpha \gg \ll \zeta_\alpha + \frac{a + ibp_\alpha}{a - ibp_\alpha} \zeta^{-1} \gg g,$$  \hspace{1cm} (3.11a)

where

$$g = A^T_1 \zeta^\infty + B^T_1 \varepsilon^\infty,$$

$$\zeta^\infty = \begin{bmatrix} \sigma^\infty_{12} \\ \sigma^\infty_{22} \\ \sigma^\infty_{32} \end{bmatrix}, \quad \varepsilon^\infty = \begin{bmatrix} \varepsilon^\infty_{11} \\ \varepsilon^\infty_{12} \\ \varepsilon^\infty_{13} \end{bmatrix}.$$  \hspace{1cm} (3.11b)

$\sigma^\infty_{ij}, \varepsilon^\infty_{ij}$ are the constant stresses and strains induced by the uniform loading applied at infinity. An alternative choice may be provided by

$$f_\alpha(\zeta) = \ll \zeta_\alpha \gg g_\alpha,$$  \hspace{1cm} (3.11c)

where

$$g_\alpha = \frac{1}{2} \ll a - ibp_\alpha \gg g,$$  \hspace{1cm} (3.11d)

and $g$ is the same as (3.11b). The infinity loading conditions are satisfied for both of the choices. The one given in (3.11c) is not a solution for uniform stress distribution,
which can be seen from the transformation function (3.2). However, in calculation (3.11c) is more convenient than (3.11a), because the singular points of (3.11c) is at infinity only while singularities occur at zero and infinity for (3.11a). In summary, any type of \( f_o \) which can represent the singular behavior including the point at infinity are all the proper choices.

Based upon the above discussion, we know that if all the singular points of the physical domain \( z_\alpha \) are considered to be located in the matrix, for different choices the complex function \( f_o \) associated with the general loading conditions may be expressed as follows.

(i) By Taylor’s expansion,

\[
f_o(\zeta) = \sum_{k=0}^{\infty} \zeta_k \zeta^k, \quad (3.12a)
\]

where

\[
\zeta_k = \frac{f_o^{(k)}(0)}{k!} = \frac{1}{2\pi i} \int_C \frac{f_o(\zeta)}{\zeta^{k+1}} d\zeta. \quad (3.12b)
\]

\( \zeta \) belongs to a bounded region where \( f_o \) is holomorphic. The cases of \( f_o = \ll \log(\zeta_\alpha - \zeta_\alpha) \gg \gamma \) and \( f_o = \ll \zeta, \gg \gamma \) belong to this category.

(ii) By Laurent’s expansion,

\[
f_o(\zeta) = \sum_{k=-\infty}^{\infty} \zeta_k \zeta^k, \quad (3.13a)
\]

where

\[
\zeta_{-k} = \frac{\zeta_k}{\zeta_k}, \quad \zeta_k = \frac{1}{2\pi i} \int_C \frac{f_o(\zeta)}{\zeta^{k+1}} d\zeta. \quad (3.13b)
\]

\( \zeta \) belongs to an annual ring where \( f_o \) is holomorphic. The cases of \( f_o = \ll \log(z_\alpha - \zeta_\alpha) \gg \gamma \) and \( f_o = \ll \zeta, \gg \gamma \) belong to this category.
For the case that all the singular points of the physical domain \( z_\alpha \) are located in the inclusions, similar approaches as those described in section 3.1 and 3.2 can be applied.

3.3 General Solutions

If the inclusion and the matrix are assumed to be perfectly bonded along the interface, the displacements and surface tractions at the interface should be continuous. That is,

\[
\begin{align*}
\bar{\gamma}_1 &= \bar{\gamma}_2 \\
\tilde{\phi}_1 &= \tilde{\phi}_2
\end{align*}
\]
where the second equation of (3.14) comes from the relation of (2.12). By using the general solution given in (3.5) and the expression given in (3.6), the traction boundary condition of (3.14) leads to

\[
\begin{align*}
B_1 \bar{f}_1(\sigma) + \bar{B}_1 \bar{f}_0(\sigma) - \sum_{k=1}^{\infty} \{ \bar{B}_2 \bar{\xi}_k + B_2 \Gamma_k \bar{\zeta}_k \} \sigma^{-k} \\
= -\bar{B}_1 \bar{f}_1(\sigma) - B_1 \bar{f}_0(\sigma) + \sum_{k=1}^{\infty} \{ B_2 \zeta_k + \bar{B}_2 \Gamma_k \bar{\zeta}_k \} \sigma^k.
\end{align*}
\] (3.15)

One of the important properties of holomorphic functions used in the method of analytic continuation is that if \( \bar{f}(\zeta) \) is holomorphic in \( S_1 \) (or \( S_2 + S_0 \)), then \( \bar{f}(1/\zeta) \) is holomorphic in \( S_2 + S_0 \) (or \( S_1 \)). Here, \( S_0 \) denotes the region inside the circle of radius \( \sqrt{m_0} \). From this property and equation (3.15), we may introduce a function which is holomorphic in the entire domain including the interface boundary, i.e.,

\[
\begin{align*}
\tilde{\theta}(\zeta) &= \begin{cases} \\
B_1 \bar{f}_1(\zeta) + \bar{B}_1 \bar{f}_0(1/\zeta) - \sum_{k=1}^{\infty} \{ \bar{B}_2 \bar{\xi}_k + B_2 \Gamma_k \bar{\zeta}_k \} \zeta^{-k}, & \zeta \in S_1 \\
-\bar{B}_1 \bar{f}_1(1/\zeta) - B_1 \bar{f}_0(\zeta) + \sum_{k=1}^{\infty} \{ B_2 \zeta_k + \bar{B}_2 \Gamma_k \bar{\zeta}_k \} \zeta^k, & \zeta \in S_2 + S_0
\end{cases}
\end{align*}
\] (3.16)
In the above, the singular points of $f_\omega$ is assumed to be located in the matrix only, i.e., the case (i) given in (3.12). Since $\tilde{\varphi}(\zeta)$ is now holomorphic and single-valued in the whole plane including the point at infinity, by Liouville’s theorem we have $\tilde{\varphi}(\zeta) \equiv \text{constant}$. However, constant function $\tilde{f}$ corresponds to rigid body motion which may be neglected. Therefore,

$$\tilde{\varphi}(\zeta) \equiv 0.$$  \hfill (3.17)

Combining (3.16) and (3.17), we have

$$\sum_{k=1}^{\infty} \{B_2 \zeta_k + B_2 \Gamma_1 \zeta_k \} \zeta^{-k} = B_1 \tilde{f}_1(\zeta) + \overline{B_1} \tilde{f}_o(1/\zeta), \quad \zeta \in S_1,$n

$$\sum_{k=1}^{\infty} \{B_2 \zeta_k + \overline{B_2} \overline{\Gamma}_1 \zeta_k \} \zeta^k = \overline{B_1} \tilde{f}_1(1/\zeta) + B_1 \tilde{f}_o(\zeta), \quad \zeta \in S_2 + S_o.$$  \hfill (3.18)

Similarly, the boundary condition $u_1 = u_2$ provides

$$\sum_{k=1}^{\infty} \{A_2 \zeta_k + A_2 \Gamma_1 \zeta_k \} \zeta^{-k} = A_1 \tilde{f}_1(\zeta) + \overline{A_1} \tilde{f}_o(1/\zeta), \quad \zeta_2 \in S_1,$n

$$\sum_{k=1}^{\infty} \{A_2 \zeta_k + \overline{A_2} \overline{\Gamma}_1 \zeta_k \} \zeta^k = \overline{A_1} \tilde{f}_1(1/\zeta) + A_1 \tilde{f}_o(\zeta), \quad \zeta_2 \in S_2 + S_o.$$  \hfill (3.19)

Cancellation of $\tilde{f}_1(\zeta)$ between (3.18) and (3.19) leads to

$$\tilde{f}_o(\zeta) = \frac{1}{2} \sum_{k=1}^{\infty} A_1^{-1} H_1 \{ (M_1 + M_2) A_2 \zeta_k + (M_1 - M_2) \overline{A_2} \overline{\Gamma}_1 \zeta_k \} \zeta^k.$$  \hfill (3.20)

where $M_k$ is the impedance matrix (Ingebrigtsen and Tonning, 1969) defined as

$$M_k = -iB_k A_k^{-1} = H_k^{-1}(I + i\zeta_k),$$  \hfill (3.21a)

or

$$M_k^{-1} = iA_k B_k^{-1} = L_k^{-1}(I + i\zeta_k^T), \quad k = 1, 2.$$  \hfill (3.21b)

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The second equalities of (3.21a) and (3.21b) have been given in (2.30) and $\zeta_k$, $\zeta_k^*$, $L_k$ are real matrices defined in (2.19) or (2.32). By substituting (3.12) into (3.20) and comparing the coefficients of corresponding terms, the unknown constants $\zeta_k$ are determined as

$$\zeta_k = \left\{ G^o - G_k G^o G_k^{-1} G_k \right\}^{-1} \left\{ t^*_k - G_k G^o G_k^{-1} t^*_k \right\}, \quad k = 1, 2, \ldots \infty \quad (3.22a)$$

where

$$G^o = (M_1 + M_2) A_2,$$

$$G_k = (M_1 - M_2) A_2 \Gamma^k,$$

$$t^*_k = -i A_1^{-T} e_k.$$

Note that the solutions associated with $\zeta^o$ are ignored because the constant stress function does not produce stress, which represents the rigid body motion. Having the solution of $\zeta_k$, function $f_1(\zeta)$ can now be obtained from (3.18) or (3.19) with the understanding that the subscripts of $\zeta$ in (3.18) or (3.19) are dropped. Once the solution of $f_1(\zeta)$ is obtained from (3.18) or (3.19), a replacement of $\zeta_1, \zeta_2$ or $\zeta_3$ should be made for each component function. This calculation procedure will be applied throughout this report. The whole field solution can then be found by using equation (3.5).

If one is interested in the interfacial stresses along the inclusion boundary, calculation may be performed by using the field solution of the matrix or inclusion. The stress components based upon the coordinate system $(\eta, \mu)$ which are, respectively, the unit vectors tangent and normal to the interface boundary, are obtained as (Hwu and Ting, 1989)
\[ \sigma_{mn} = \eta_{\theta}^T \phi_{,n}, \quad \sigma_{nn} = \eta_{\theta}^T \phi_{,n}, \quad \sigma_{m3} = (\phi_{,n})_3, \]
\[ \sigma_{nn} = -\eta_{\theta}^T \phi_{,m}, \quad \sigma_{nm} = -\eta_{\theta}^T \phi_{,m} = \sigma_{mn}, \quad \sigma_{n3} = -(\phi_{,m})_3, \]
\[ \text{(3.23a)} \]

where
\[ \eta_{\theta}^T(\theta) = (\cos \theta, \sin \theta, 0), \quad \eta_{\theta}^T(\theta) = (-\sin \theta, \cos \theta, 0), \]
\[ \text{(3.23b)} \]

and the angle \( \theta \) is directed counterclockwise from the positive \( x_1 \)-axis to the direction of \( \eta_{\theta} \). The relation between \( \theta \) and \( \psi \) is
\[ \rho \cos \theta = a \sin \psi, \quad \rho \sin \theta = -b \cos \psi, \]
\[ \rho = \sqrt{a^2 \sin^2 \psi + b^2 \cos^2 \psi}. \]
\[ \text{(3.24)} \]

The derivative of \( \phi \) along the interface, \( \phi_{,n} \), should be continuous across the interface since \( \phi_1 = \phi_2 \) along the interface boundary. However, \( \phi_{,m} \) may be discontinuous. The evaluation of \( \phi_{,m} \) and \( \phi_{,n} \) can be performed by using chain rule as
\[ \frac{\partial f}{\partial m} = \frac{\partial f}{\partial \zeta_\alpha} \frac{\partial \zeta_\alpha}{\partial \psi} \frac{\partial \psi}{\partial z_\alpha} \left[ \frac{\partial z_\alpha}{\partial x_1} \frac{\partial x_1}{\partial m} + \frac{\partial z_\alpha}{\partial x_2} \frac{\partial x_2}{\partial m} \right], \]
\[ \frac{\partial f}{\partial n} = \frac{\partial f}{\partial \zeta_\alpha} \frac{\partial \zeta_\alpha}{\partial \psi} \frac{\partial \psi}{\partial z_\alpha} \left[ \frac{\partial z_\alpha}{\partial x_1} \frac{\partial x_1}{\partial n} + \frac{\partial z_\alpha}{\partial x_2} \frac{\partial x_2}{\partial n} \right], \]
\[ \text{(3.25a)} \]

where
\[ \zeta_\alpha = e^{i \psi}, \quad \frac{\partial \zeta_\alpha}{\partial \psi} = ie^{i \psi}, \quad \frac{\partial z_\alpha}{\partial \psi} = -\rho (\cos \theta + p_a \sin \theta), \]
\[ \frac{\partial x_1}{\partial m} = -\sin \theta, \quad \frac{\partial x_2}{\partial m} = \cos \theta, \quad \frac{\partial x_1}{\partial n} = \cos \theta, \quad \frac{\partial x_2}{\partial n} = \sin \theta, \]
\[ \text{(3.25b)} \]
\[ \frac{\partial z_\alpha}{\partial x_1} = 1, \quad \frac{\partial z_\alpha}{\partial x_2} = p_a. \]
along the interface boundary. If the field solution of the inclusion given in (3.5b) with \( f_2(\zeta^*) \) obtained in (3.6) and (3.22) is used, by applying (3.25) we have

\[
\hat{\varphi}_{2,m} = -\sum_{k=1}^{\infty} \frac{2k}{\rho} \text{Im} \{ B_2 \ll p_0^*(\theta) \gg [e^{-ik\psi} \Gamma_k^{*} - e^{ik\psi} \Gamma_k] \zeta_k \},
\]

\[
\hat{\varphi}_{1,n} = \hat{\varphi}_{2,n} = \hat{\varphi}_n = -\sum_{k=1}^{\infty} \frac{2k}{\rho} \text{Im} \{ B_2 [e^{-ik\psi} \Gamma_k^{*} - e^{ik\psi} \Gamma_k] \zeta_k \},
\]

where \( \text{Im} \) stands for the imaginary parts. Similarly, \( \hat{\varphi}_{1,m} \) is obtained by applying the field solution of the matrix given in (2.5a), or by (Ting and Yan, 1991)

\[
\begin{pmatrix}
\bar{\varphi}_{1,m} \\
\hat{\varphi}_{1,m}
\end{pmatrix} = N(\theta) \begin{pmatrix}
\bar{u}_{1,n} \\
\hat{u}_{1,n}
\end{pmatrix},
\]

in which \( \bar{u}_{1,n} \) and \( \hat{u}_{1,n} \) can be obtained by using the field solution of the inclusion or matrix since they are continuous across the interface.

If the function \( f_0 \) corresponding to the unperturbed elastic field is chosen to be the case (ii) given in (3.13), equation (3.15) may be replaced by

\[
B_1 \bar{f}_1(\sigma) + \sum_{k=1}^{\infty} \left\{ B_1 \bar{\xi}_k + B_1 \bar{\Gamma}_k \zeta_k - B_2 \bar{z}_k - B_2 \Gamma_k \zeta_k \right\} \sigma^{-k} = -B_1 \bar{f}_1(\sigma) - \sum_{k=1}^{\infty} \left\{ B_1 \xi_k + \bar{B}_1 \Gamma_k \bar{\zeta}_k - \bar{B}_2 \bar{\zeta}_k - \bar{B}_2 \Gamma_k \bar{\zeta}_k \right\} \sigma^k.
\]

By the method of analytic continuation shown previously for case (i), one may find that the solution of \( \zeta_k \) for this case has exactly the same expression as equation (3.22), and function \( \bar{f}_1(\zeta) \) is obtained as

\[
\bar{f}_1(\zeta) = -\sum_{k=1}^{\infty} B_1^{-1} \left\{ B_1 \bar{\xi}_k + B_1 \bar{\Gamma}_k \bar{\zeta}_k - B_2 \bar{z}_k - B_2 \Gamma_k \bar{\zeta}_k \right\} \zeta^{-k},
\]

or

\[
\bar{f}_1(\zeta) = -\sum_{k=1}^{\infty} A_1^{-1} \left\{ A_1 \xi_k + A_1 \Gamma_k \zeta_k - A_2 \bar{z}_k - A_2 \Gamma_k \bar{\zeta}_k \right\} \zeta^{-k}.
\]
Notice again that $f_k, k = 0, 1, 2,$ are required to have the form of $\{f_1(\xi) \ f_2(\xi) \ f_3(\xi)\}_k^T$. The expressions for the interfacial stresses are also the same as case (i).

3.4 Examples

3.4.1 Point forces in the matrix

Consider an infinite anisotropic medium containing an elastic inclusion, subjected to a point force applied on $\zeta_\alpha = \xi_\alpha$ located in the matrix. The complex function $f_{\alpha}(\xi)$ associated with the unperturbed elastic field is represented by the function given in (3.10). The simplest condition of the elastic inclusion problem is that the matrix and inclusion are composed of the same material. Our results for the elastic inclusions should therefore be checked by this simplest case. When $A_1 = A_2 = \zeta, M_1 = M_2 = \zeta$, by equation (3.22), we have

$$G_\alpha = 2H^{-1}A = -i\zeta^{-T}, \quad G_k = 0, \quad \zeta_k = \zeta_k, \quad k = 1, 2, \ldots \infty, \quad (3.30a)$$

where $\zeta_k$ can be evaluated by using (3.10) and (3.12b), as

$$\zeta_k = \zeta_k - k, \quad (3.30b)$$

With the use of (3.18)$_1$, (3.12a) and (3.30), we obtain

$$f_{j1}(\zeta) = \sum_{k=1}^{\infty} \Gamma_k \zeta_k^{-k}, \quad (3.31a)$$

The function $f_{j2}(\zeta)$ corresponding to the perturbative field of inclusions is obtained by using (3.6) and (3.30a), which is

$$f_{j2}(\zeta) = \sum_{k=1}^{\infty} (\zeta_k^k + \Gamma_k \zeta^k \zeta_k^{-k}), \quad (3.31b)$$

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The infinite series representation of $f_1$ and $f_2$ can be shown to be a Taylor's expansion of logarithmic function, i.e.,

$$\sum_{k=1}^{\infty} \gamma_k \xi_k \zeta^{-k} = \ll \log \left(1 - \frac{(\zeta \alpha)^2}{\zeta \alpha} \right) \gg \eta , \quad \zeta \in S_1 + S_2,$$

$$\sum_{k=1}^{\infty} \xi_k \zeta^k = \ll \log \left(1 - \frac{\zeta \alpha}{\zeta \alpha} \right) \gg \eta , \quad \zeta \in S_2. \quad (3.31c)$$

Combining the results given in (3.10) and (3.31), one may prove that

$$f_0(\zeta) + f_1(\zeta) = f_2(\zeta) = \ll \log (\zeta \alpha - \zeta \alpha) \gg \eta + \text{constant}$$

which is the solution for a homogeneous medium under point forces.

Since the analytical solution is not available in the literature for the general elastic inclusions subjected to a concentrated force located in the matrix, a series of numerical results are presented. When the inclusions are very soft or hard, the values of the problems with holes or rigid inclusions can be used as a check for our results. Note that all the results with regard to the problems of holes and rigid inclusions shown in this section, quoted as a closed form solution, will be present in Chapters 4 and 5.

Consider both of the matrix and inclusion are orthotropic materials. The material properties of the matrix are

$$E_1 = 144.8 \text{ Gpa}, \quad E_2 = E_3 = 10.7 \text{ Gpa}$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.31$$

$$G_{12} = G_{13} = G_{23} = 4.5 \text{ Gpa}$$

while the properties of the inclusion are assumed to be proportional to the matrix as

$$k = \frac{(E_i)^2}{(E_i)_1} = \frac{(G_{ij})^2}{(G_{ij})_1} , \quad i, j = 1, 2, 3,$$
\[ \nu_{12} = \nu_{13} = \nu_{23} = 0.31 \]

where \( k \) is the index of hardness (or softness). When \( k > 1 \) the inclusion is harder than the matrix, while \( k < 1 \) means soft. A rigid inclusion or hole can therefore be approximated by letting \( k \to \infty \) or \( k \to 0 \). We now consider a concentrated force \( \hat{p}(\text{Nt/m}) \) directed in the \( x_2 \)-axis applied on the point \( (\hat{x}_1/a, \hat{x}_2/a) = (0,5) \).

The elliptic inclusion is represented by \( b/a = 0.75 \). The hoop stress \( \sigma_{nn} \) of elastic inclusion for various \( k \) are calculated by (3.23), (3.26), and (3.27). The results for the normalized hoop stress, \( \sigma_{nn}a/\hat{p} \), are shown in Table 3.1, from which we see that the solutions for rigid inclusions or holes are approximated by \( k = 10^6 \) or \( k = 10^{-6} \). The correctness of our results is therefore verified. To see the effect of elliptic shape and the singular behavior near the crack tips or the tips of rigid line inclusions, a series of numerical data for the hoop stress at \( \psi = 0^\circ \), \( \sigma_{nn}^0 \), are shown in Figures 3.3 and 3.4. A nearly constant value of the hoop stress for \( b \to 0 \) is achieved when the inclusion is not a hole or rigid medium, which means that no singular behavior occurs for the general elastic inclusions. For elliptic holes or rigid inclusions, singular behavior occurs when \( b \to 0 \) which is expected for the cracks and rigid line inclusions.

3.4.2 Uniform loads at infinity

In the case when the elastic inclusion in an infinite matrix is subjected to a uniform load at infinity, detail analysis has been given in Hwu and Ting (1989) by using semi-inverse method, i.e., the function form of \( \mathcal{f}(\zeta) \) is chosen before calculation. In this report, without any prior choices, general solutions of \( \mathcal{f}(\zeta) \) are obtained for arbitrary loading conditions. In order to verify this solution, we reduce our results
to uniform loading condition since it is the only analytical solution available for the general elastic inclusion problems.

The complex function \( f_o(\zeta) \) is chosen as those shown in (3.11a), which belongs to case (ii). Same as the case of point force, we first check the condition when the matrix and inclusion are composed of the same material. If \( A_1 = A_2, B_1 = B_2, \ M_1 = M_2 \), we have, by (3.22) and (3.29),

\[
\begin{align*}
\xi_1 &= \xi_1, \quad \xi_k = 0, \quad k = 2, 3, \ldots \infty, \\
\tilde{\xi}_1 &= 0.
\end{align*}
\] (3.32)

The zero perturbed solution means that there is no inclusion effect for the homogeneous medium, which is expected since the \( \tilde{f}_o \) chosen represents the exact solution of homogenous medium subjected to uniform loading at infinity.

For general elastic inclusions, the functions \( \tilde{f}_1 \) and \( \tilde{f}_2 \) corresponding to the perturbed fields of matrix and inclusion are obtained from (3.22), (3.29), and (3.6) as

\[
\begin{align*}
\tilde{f}_1(\zeta) &= -\llb \xi_1^{-1} \rrb_B^{-1} \left\{ B_1 \xi_1 + B_1 \Gamma_1 \xi_1 - B_2 \Gamma_1 \xi_1 - B_2 \Gamma_1 \xi_1 \right\}, \\
\tilde{f}_2(\zeta) &= \llb \frac{2x_\alpha^*}{a - ibp_\alpha} \rrb \xi_1,
\end{align*}
\] (3.33)

where

\[
\xi_1 = -i \left\{ \bar{G}_o - \bar{G}_1 \bar{G}_o^{-1} G_1 \right\}^{-1} \left\{ A_1^T \xi_1 + \bar{G}_1 \bar{G}_o^{-1} A_1^T \xi_1 \right\},
\]

and

\[
\xi_1 = \frac{1}{2} \llb a - ibp_\alpha \rrb (A_1^T \xi_1 + B_1^T \xi_1). 
\]

Note that \( \tilde{f}_2 \) obtained in (3.33)_2 represents a state of uniform stress which has been observed by Eshelby (1957). By numerical calculation shown in Figures 3.5
and 3.6, we may prove that this solution is identical to those given in Lekhnitskii (1968) for circular inclusion, and Hwu and Ting (1989) for elliptic inclusion, in which the material properties of the matrix, plywood, are

\[ E_1 = 11.8 \text{ Gpa}, \quad E_2 = 5.9 \text{ Gpa}, \quad G_{12} = 0.69 \text{ Gpa}, \quad \nu_{12} = 0.071, \]

and the hardness index \( k = 2 \).

### 3.4.3 Interactions between dislocations and elastic inclusions

Interactions between dislocations and inhomogeneities have been a topic of considerable research. Greater understanding of material defects can be gained through the solution of suitable elasticity problems. Some researchers investigated the interactions between dislocations and elliptic inclusions or holes for isotropic materials (Stagni and Lizzio, 1983 and Santare and Keer, 1986). However, the concept of an elastically isotropic crystal is an idealization; all real crystals are anisotropic. Isotropic theory may lead to useful results, but for some cases it is an inadequate approximation. Hence, it is necessary to analyze such interaction problems by using anisotropic theory.

To study the dislocation interaction problems, a common measurement is the image force which is a virtual force representing the change in the free energy of the system with displacement of the dislocation. For the dislocation with Burgers vector \( \mathbf{b} \) located on \( \Sigma \), the total stress fields can be obtained in a straight-forward manner from the known solution for a point force by using a certain analogy between dislocations and point forces. If the total stress field is split into two parts \( \sigma^{d}_{ij} \) and \( \sigma_{ij} \) where \( \sigma^{d}_{ij} \) is the self-stress of the dislocation in an infinite homogeneous medium,
only the stress \( \sigma_{ij} \) induced by the presence of inclusions does work on the slip plane as the dislocation moves (Hirth and Lothe, 1982). Consider a dislocation with Burger vector \( \hat{b} \) located at the points \((\hat{x}_1, \hat{x}_2)\). By visualising the cut parallel to \( x_1 \) plane from \( x_1 = \hat{x}_1 \) to \( x_1 = \infty \) and giving the surface of the cut a relative displacement \( \hat{b} \). The work done in the process against the stresses of the defect such as inclusions or holes i.e., the interaction energy \( E^i \) is (Stroh, 1958)

\[
E^i = \frac{1}{2} \int_{\hat{x}_1}^{\infty} (\hat{b}_1 \sigma_{21}^i + \hat{b}_2 \sigma_{22}^i + \hat{b}_3 \sigma_{23}^i) \, dx_1.
\]  

(3.34)

Since \( \partial \hat{\phi}^i / \partial x_1 = \{\sigma_{21}^i, \sigma_{22}^i, \sigma_{23}^i\}^T \), it follows that

\[
E^i = -\hat{b}^T \hat{\phi}^i(\hat{x}_1, \hat{x}_2),
\]  

(3.35)

where \( \hat{\phi}^i(\infty) \) is assumed to be zero. If the dislocation moves along \( s \) direction, the generalized force \( F^i \) in the \( s \) direction on the dislocation is defined as the negative gradient of the interaction energy, i.e.,

\[
F^i = -\frac{\partial E^i}{\partial s}.
\]  

(3.36)

The calculation of total stress function \( = \hat{\phi}^i + \hat{\phi}^d \) for the dislocation in a general position with respect to the inclusion is similar to the cases of point forces discussed in Section 3.4.1. The only difference is that \( \mathcal{Q} = \mathcal{A}^T \hat{b} / 2\pi i \) is now replaced by \( \mathcal{Q} = \mathcal{B}^T \hat{b} / 2\pi i \). To evaluate \( \hat{\phi}^i \), a substraction of the self-stress \( \hat{\phi}^d \) from the total stress should be made. Note that \( \hat{\phi}^d \) is equal to

\[
\hat{\phi}^d = 2\text{Re}\{\mathcal{B} \ll \log(z_\alpha - \hat{z}_\alpha) \gg \mathcal{B}^T \hat{b} / 2\pi i}\},
\]  

(3.37)

which is different from that obtained by \( f_\alpha \) given in (3.10). The total stress function of elastic inclusion with a dislocation is obtained from (3.5a), in which (3.10),
(3.18)_1, (3.22a) and (3.30b) are used. Using the total stress function, (3.35), (3.37), and the relation

$$\log(\zeta_\alpha - \tilde{\zeta}_\alpha) = \log(\bar{\zeta}_\alpha - \tilde{\zeta}_\alpha) + \log\left(1 - \frac{\zeta_\alpha^2}{\bar{\zeta}_\alpha \tilde{\zeta}_\alpha}\right),$$

(3.38)

where the critical point $\zeta_\alpha$ has been given in (3.3), we can obtain the interaction energy in an explicit form as

$$E^i = -\frac{1}{2}\hat{\Phi}\nabla\tilde{b},$$

(3.39)

where

$$\hat{\Phi} = \frac{1}{\pi} Im\left\{B_1 \ll \log\left(1 - \frac{\zeta_\alpha^2}{\bar{\zeta}_\alpha \tilde{\zeta}_\alpha}\right) \gg B_1^T\right\} + \frac{1}{\pi} Im\left\{B_1 \sum_{k=1}^{\infty} \ll \zeta_\alpha^{-k} \gg B_1^{-1}\tilde{D}_k\right\}$$

$$+ \frac{1}{\pi} Im\left\{B_1 \sum_{j=1}^{3} \ll \log(\zeta_\alpha^{-1} - \tilde{\zeta}_j) \gg B_1^{-1}\bar{B}_j I_j B_1^T\right\},$$

(3.40a)

$$\tilde{D}_k = \frac{i}{k} \left(\bar{B}_j \bar{D}_k - B_2 \bar{\Gamma}_k \bar{D}_k \bar{E}_k\right) A_k^{-T} \ll \tilde{\zeta}_\alpha^{-k} \gg B_1^T$$

$$+ \frac{i}{k} \left(B_2 \Gamma_k \bar{D}_k - \bar{B}_2 \bar{\Gamma}_k \bar{E}_k\right) A_k^{-T} \ll \tilde{\zeta}_\alpha^{-k} \gg B_1^T,$$

(3.40b)

$$D_k = \left\{G_k - \bar{G}_k \bar{G}_k^{-1} G_k\right\}^{-1},$$

$$E_k = \bar{G}_k \bar{G}_k^{-1},$$

(3.40c)

and

$$I_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

(3.41)

In the following, we consider an edge dislocation with Burgers vector $\tilde{\hat{b}} = (\hat{b}_1, 0, 0)$ directed along $x_1$-axis so that the slip plane is parallel to the plane $x_2 = 0$. The glide component of the image force per unit length tending to move the
dislocation towards or away from the $x_2$-axis may be found from (3.35) and (3.36), which is

$$F_1 = \sigma_{12}^i \hat{b}_1.$$ 

The interactions between dislocations and inclusions are then shown by the contour of the glide component of the image force per unit length ($F_1$). The force is nondimensionalized by dividing $L_{11} b_1 / 4\pi$. Figures 3.7-3.9 show that the contour diagrams for various shapes of elastic inclusion interacted with dislocations, in which the material properties are the same as those given in Section 3.4.2 and the soft and hard inclusions are represented by $k=0.5$ and 2, respectively. From these results, we observe that the glide component of image force is positive for hard inclusion and is negative for soft inclusion. In regions where the glide components of the image force are negative, the dislocation is attracted to the $x_2$-axis. A positive value indicates that the dislocation is repelled from the $x_2$-axis. Therefore, when the glide force is zero, the position can be thought of as an equilibrium position for the dislocation. In all the plots, the $x_2$-axis is a line of zero glide force.
CHAPTER IV

HOLES

In this chapter, an anisotropic medium with an elliptic hole under arbitrary loading is considered. Since only the matrix is considered, the subscripts 1 and 2 used to distinguish the matrix and inclusion are omitted for simplicity. The general solutions in terms of the variables $\zeta_{\alpha}$ given in (3.5a) becomes

$$u = A [f_{\alpha}(\zeta) + \overline{f}(\zeta)] + \overline{A} [\overline{f}_{\alpha}(\zeta) + \overline{f}(\zeta)]$$

$$\phi = B [f_{\alpha}(\zeta) + \overline{f}(\zeta)] + \overline{B} [\overline{f}_{\alpha}(\zeta) + \overline{f}(\zeta)].$$

When the inclusion is a traction-free hole, $\phi = \zeta$ along the hole boundary, which leads to

$$B f_{\lambda}(\sigma) + \overline{B} f_{\lambda}(\sigma) = - \overline{B} f_{\lambda}(\sigma) - B f_{\lambda}(\sigma)$$

if $f_{\alpha}$ belongs to case (i). By the method of analytic continuation we find that

$$\overline{f}(\zeta) = - B^{-1} \overline{B} f_{\lambda}(1) \zeta.$$  \hfill (4.3)

By a way similar to those shown in (3.23)-(3.27), the hoop stress $\sigma_{nn}$ along the hole boundary is obtained as

$$\sigma_{nn} = -n^T(\theta) \hat{\phi},m,$$  \hfill (4.4a)

where

$$\hat{\phi},m = - \frac{4}{\rho} N_3(\theta) L^{-1} Re\{\sigma B f_{\lambda}'(\sigma)\},$$  \hfill (4.4b)
or
\[
\tilde{\phi}_{m} = -\frac{4}{\rho} N_{3}(\theta) \tilde{L}^{-1} \sum_{k=1}^{\infty} \text{Re}\{ke^{ik\psi}B\tilde{\zeta}_{k}\},
\] (4.4c)
when \( f_{\phi} \) is expressed by the Taylor's expansion as (3.12). During the derivation of equation (4.4), one should be very careful about the \( f_{\phi}(\zeta) \) given in (4.3), whose argument of each component function should be replaced by \( \zeta_{1}, \zeta_{2} \) and \( \zeta_{3} \), respectively. Moreover, the third identity in (2.31) has been used. If \( f_{\phi} \) belongs to case (ii), similar approach can be applied and the results are
\[
f_{\phi}(\zeta) = -B^{-1} \sum_{k=1}^{\infty} \{B\tilde{\zeta}_{k} + B\tilde{\zeta}_{k} \} \zeta^{-k}. \] (4.5)

The expression for the hoop stress is the same as (4.4a) and (4.4c).

4.1 Point forces

When the medium is subjected to a point force \( \tilde{p} \) located at \( \zeta = (\tilde{x}_{1}, \tilde{x}_{2}) \), the complex function \( f_{\phi}(\zeta) \) can be written explicitly by substituting (3.10) into (4.3) with the understanding that the subscripts of \( \zeta \) are dropped before the multiplication of matrices and a replacement of \( \zeta_{\alpha} \) should be made for each component function of \( f_{\phi}(\zeta) \) after the multiplication of matrices. The result is
\[
f_{\phi}(\zeta) = \sum_{k=1}^{3} \ll \log(\zeta_{\alpha}^{-1} - \tilde{\zeta}_{k}) \gg B^{-1} B\tilde{I}_{k} A^{T} \tilde{p} / 2\pi i \] (4.6)
where \( \tilde{I}_{k} \) are the same as those given in (3.40). The derivative \( \tilde{\phi}_{m} \) shown in (4.4b) used to calculate the hoop stress can be reduced to
\[
\tilde{\phi}_{m} = \frac{2}{\pi \rho} N_{3}(\theta) \tilde{L}^{-1} \text{Re}\{B \ll e^{i\psi} (e^{i\psi} - \tilde{\zeta}_{\alpha})^{-1} \gg A^{T}\} \tilde{p}. \] (4.7)

The numerical results of the hoop stress are shown in Table 3.1 as a check for the problem of elastic inclusions.
4.2 Uniform loads at infinity

If the complex function \( f_0(\zeta) \) associated with the unperturbed elastic field is chosen as those shown in (3.11a), it belongs to the case (ii). The function \( f_\zeta(\zeta) \) corresponding to the perturbative field of matrix is then obtained from (4.5) with

\[
\zeta_1 = \frac{1}{2} \ll a - ib\alpha \gg q,
\]

\[
\zeta_k = 0, \quad k = 2, 3, \ldots \infty.
\]

The final simplified result is

\[
f_\zeta(\zeta) = -\frac{1}{2} \ll \zeta_1^{-1} \gg B^{-1}(a t_2^\infty - ib t_1^\infty),
\]

which can be proved to be identical to those given in Hwu and Ting (1989). In the derivation of (4.9), the first and third identities given in (2.29) with \( \theta = 0 \), have been used. Moreover, the identity (Ting, 1988a)

\[
N_1^T t_2^\infty + N_3 \zeta_1^\infty = -t_1^\infty,
\]

and (2.19) are also needed.

As stated in Section 3.2, \( f_0 \) can also be chosen as

\[
f_0(\zeta) = \ll \zeta \gg q_0, \quad q_0 = \frac{1}{2} \ll a - ib\alpha \gg q.
\]

For this choice, function \( f_0(\zeta) \) should be found by using (4.3) instead of (4.5) since \( f_0(\zeta) \) now belongs to the case (i). By careful derivation, one can prove that the final results of \( f_0 + f_1 \) are the same for different choices of \( f_0 \). A real form solution for the hoop stress along the hole boundary can be obtained by substituting (4.8) into (4.4c) and applying the identities given in (2.19), (2.29), (4.10), which can also be proved to be identical to those shown in Hwu and Ting (1989).
4.3 Interactions between dislocations and holes

Similar to the procedures described in 3.4.4, the interaction energy can be obtained in an explicit form as

\[
E^i = -\hat{\overrightarrow{b}} = \frac{1}{\pi} Im \left\{ B_1 \ll \log \left( 1 - \frac{\overrightarrow{c}^2}{\overrightarrow{\zeta} \overrightarrow{\zeta}} \right) \gg \overrightarrow{B_1^T} \right\} \hat{\overrightarrow{b}}
\]

\[
-\overrightarrow{b} = \frac{1}{\pi} Im \left\{ B_1 \sum_{j=1}^{3} \ll \log (\overrightarrow{\zeta}^{-1} - \overrightarrow{\zeta}_j) \gg B_1^{-1} \overrightarrow{B_1^T} \right\} \overrightarrow{b}.
\]

(4.11)

Consider an edge dislocation with Burgers vector \( \hat{\overrightarrow{b}} = (\hat{b}_1, 0, 0) \) directed along \( z_1 \)-axis. For isotropic material with \( \nu = 0.1 \), the results for the contour of the glide component of the image force per unit length \( (F_1) \), nondimensionalized by dividing \( L_1 \hat{b}_1/4\pi \), are shown in Figures 4.1-4.3, which are same as those given by Stagni and Lizzo (1983). Note that in numerical calculation, a small perturbation of the material constants for isotropic materials has been used to avoid the problem of repeated eigenvalues \( p_\alpha \). For anisotropic media with the material properties considered in section 3.4.2, the contour of the glide component of the image force per unit length \( (F_1) \) are also shown in Figure 4.1-4.3. The effect of anisotropic on the \( F_1 \) can then be studied from these figures.
CHAPTER V

RIGID INCLUSIONS

Holes are extreme cases of elastic inclusions for which the inclusion is extraordinary soft relative to the matrix. The other extreme case is rigid inclusion which means that the inclusion is absolutely rigid and can not be deformed. However, a rigid body rotation $\omega$ relative to the matrix may occur. Hence, the boundary conditions for the cases of rigid inclusions are

$$u = \frac{\omega}{2}(k\sigma + \bar{k}\sigma^{-1})$$  \hspace{1cm} (5.1a)

where

$$k = \begin{cases} 
ib \\
a \\
0 
\end{cases}$$  \hspace{1cm} (5.1b)

and $\sigma = e^{i\psi}$. Using the general solution given in (4.1) and the above boundary condition, we have

$$\mathcal{A}\mathcal{f}(\sigma) + \mathcal{A}\mathcal{f}_o(\sigma) - \frac{\omega}{2}k\sigma^{-1} = -\mathcal{A}\mathcal{f}(\sigma) - \mathcal{A}\mathcal{f}_o(\sigma) + \frac{\omega}{2}k\sigma$$  \hspace{1cm} (5.2)

if $\mathcal{f}_o$ belongs to case (i). By the method of analytic continuation, we find that

$$\mathcal{f}(\zeta) = -A^{-1}\mathcal{A}\mathcal{f}_o\left(\frac{1}{\zeta}\right) + \frac{\omega}{2\zeta}A^{-1}\bar{k}.$$  \hspace{1cm} (5.3)

To determine $\omega$, we use the condition that the total moment about the origin due to the traction $\mathcal{t}^m$ on the surface of rigid inclusion vanishes, i.e.,
\[ \int_0^{2\pi} \left\{ b \sin \psi(t_m) - a \cos \psi(t_m) \right\} \rho \, d\psi = 0. \quad (5.4) \]

By applying the relation \( t_m = \partial \varphi / \rho \partial \psi \) (Hwu and Ting, 1989) and the solutions given in (4.1) and (5.3), equation (5.4) leads to

\[ \omega = -2 \int_0^{2\pi} \gamma^T \left( A^{-T} f' \right) (e^{i\psi}) \frac{d\psi}{\pi \text{Im} \left\{ k^T \frac{B A^{-1} k}{k} \right\}}, \quad (5.5a) \]

where

\[ \gamma^T = (-b \sin \psi, a \cos \psi, 0), \quad (5.5b) \]

and prime (\( \prime \)) denotes differentiation with respect to its argument. Similar to the problems of elastic inclusions, the interfacial stresses can be determined by \( \varphi_m \) and \( \varphi_n \), which are

\[ \varphi_m = -\frac{4}{\rho} N_1^T(\theta) H^{-1} \text{Re} \left\{ e^{i\psi} A f_\sigma(\sigma) \right\} + \frac{\omega}{\rho} \text{Re} \left\{ i e^{-i\psi} B P(\theta) A^{-1} k \right\}, \quad (5.6a) \]

or

\[ \varphi_m = -\frac{4}{\rho} H^{-1} \text{Re} \left\{ e^{i\psi} A f'_\sigma(\sigma) \right\} + \frac{\omega}{\rho} \text{Re} \left\{ i e^{-i\psi} B A^{-1} k \right\}, \quad (5.6b) \]

in which the third identity of (2.31) has been used. Similarly, if \( f_\sigma \) belongs to case (ii), we have

\[ f_\sigma(\zeta) = -A^{-1} \sum_{k=1}^{\infty} \left( A_{\zeta k} + A_{\zeta k} e_k \right) \zeta^{-k} + \frac{\omega}{2 \zeta} A^{-1} k, \quad (5.7a) \]

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and
\[
\omega = \frac{-2\text{Im}\{k^T A^{-T} \bar{z}_1\}}{\text{Im}\{k^T B A^{-1} \bar{k}\}}.
\]  
\hspace{1cm} (5.7b)

The expressions of the derivatives $\phi_{,m}$ and $\phi_{,n}$ are exactly the same as (5.6b).

### 5.1 Point forces

Similar to (4.6), the explicit solution of $f_{\phi}(\zeta)$ given in (5.3) for a concentrated force $\vec{p}$ applied on $\zeta = \zeta_0$ is obtained as
\[
f_{\phi}(\zeta) = \sum_{k=1}^{3} \ll \log(\zeta_0^{-1} - \zeta_k) \gg A^{-1} A^{-1} I_k A^T \vec{p} / 2\pi i + \omega / 2 \ll \zeta_0^{-1} \gg A^{-1} \bar{\zeta}
\]  
\hspace{1cm} (5.8)

where the relative rotation $\omega$ can be evaluated by (5.5) with $f_{\phi}$ given in (3.10). With the aid of residue theorem, we obtain
\[
\omega = \frac{\text{Re}\{k^T A^{-T} \ll \zeta_0^{-1} \gg A^T \vec{p}\}}{\pi \text{Im}\{k^T B A^{-1} \bar{k}\}}.
\]  
\hspace{1cm} (5.9)

If the load is applied on the interface boundary, i.e., $\zeta_0 = e^{i\phi}$, we have
\[
\omega = \frac{-\hat{x}_2 \hat{p}_1 + \hat{x}_1 \hat{p}_2}{\pi \text{Im}\{k^T B A^{-1} \bar{k}\}}
\]  
\hspace{1cm} (5.10)

where $(\hat{x}_1, \hat{x}_2) = (a \cos \hat{\psi}, b \sin \hat{\psi})$ is the location of the applied force $\vec{p} = (\hat{p}_1, \hat{p}_2, \hat{p}_3)$. This solution of $\omega$ is equivalent to the one given by Ting and Yan (1991).

### 5.2 Uniform load at infinity

Similar to the hole problems, substitution of (4.13) into (5.7) and use of the identities given in (2.19c), (2.29)$_2$, (2.29)$_3$ and (Ting, 1988a)
\[
N_2 \zeta_2^\infty + N_1 \zeta_1^\infty = \zeta_2^\infty,
\]  
\hspace{1cm} (5.11)
provide

\[ f(\zeta) = -\frac{1}{2} \ll \frac{1}{\zeta_\alpha^2} \gg A_2^{-1}(a_2 \xi_1^\infty + ib_2 \xi_2^\infty + \omega \bar{k}) \],

\[
\omega = \frac{a^2(H^{-1} \xi_1^\infty)_2 - ab[(H^{-1} \xi_1^\infty)_1 + (H^{-1} \xi_2^\infty)_2] - b^2(H^{-1} \xi_2^\infty)_1}{a^2(H^{-1})_{22} + 2ab(H^{-1})_{21} + b^2(H^{-1})_{11}}, \quad (5.12)
\]

which can be proved to be equivalent to the one given in Hwu and Ting (1989).

5.3 Interaction between dislocation and rigid inclusions

By similar procedure outlined in 3.4.4 and use of (5.3), the interaction energy is given by

\[
E' = -\hat{b}^T \frac{1}{\pi} \text{Im} \left\{ B_1 \ll \log \left(1 - \frac{\xi^2}{\zeta_\alpha^2} \right) \gg A_1 \right\} \hat{b}^T
\]

\[
-\hat{b}^T \frac{1}{\pi} \text{Im} \left\{ \sum_{j=1}^{3} \ll \log(\xi_j^{-1} - \bar{\zeta}_j) \gg A_j^{-1} A_j^T T \right\} \hat{b}^T
\]

\[
-\omega \hat{b}^T \text{Re} \left\{ \hat{B} \ll \frac{1}{\xi_\alpha} \gg A_2^{-1} \bar{k} \right\},
\]

where

\[
\omega = \frac{\text{Re} \left\{ \bar{k}^T A^{-T} \ll \zeta_\alpha^{-1} \gg B \right\} \hat{b}}{\pi \text{Im} \left\{ \bar{k}^T B A^{-1} \hat{k} \right\}}. \quad (5.13b)
\]

If a dislocation with \( \hat{b} = (\hat{b}_1, 0, 0) \), results concerning the contour of the glide component of the imaginary force per unit \( (F_1) \), nondimensionalized by dividing \( L_{11} \hat{b}_1/4\pi \), are shown in Figures 5.1-5.3.
CHAPTER VI

CONCLUSIONS

In this dissertation, the laminated composites are modelled by an anisotropic plate. The defects include elastic inclusions, rigid inclusions, holes and cracks. The free edge effects, which are important for laminated composites and are ignored when the laminates are modelled as a homogeneous anisotropic plate, are also considered in this dissertation by treating each sublaminate as an individual anisotropic body.

A general solution for the anisotropic elastic inclusion problems is formulated by applying the Stroh’s formalism and perturbation concept. To apply for the elliptic boundary and arbitrary loading conditions, the technique of conformal mapping and a special method of analytical continuation are developed. Based upon the choices of unperturbed stress functions, two different possible conditions are considered. One is represented by the Taylor’s series, the other is represented by Laurent’s expansion. The final solutions will not be affected by the choices of unperturbed stress functions since two types can all capture singular behavior of loading. The loading considered here is only acting in the matrix. In the same way, the cases in which loading or singularities are inside the inclusion can be solved with ease. A universal solution for the stresses and deformations in the matrix and inclusion is presented in this study. The solutions include the problems of elastic inclusions, holes and rigid inclusions subjected to arbitrary loadings.

Some special loadings are solved explicitly, such as the point forces, dislocations and uniform loadings. The elasticity solutions of the point force problems can be
used as the fundamental solutions of the boundary element, in which the discretization along the inclusion or hole boundary can be avoided. Therefore, less computer time and storage, and higher accuracy can be achieved. The case of uniform loading is the one that detail and complete analysis has been provided in the literature, and is used here to verify the correctness of our results. The problems of dislocation can be obtained directly by a certain analogy between dislocations and point forces. The results are frequently used as kernel functions of integral equations to consider the interactions between inclusions and cracks. The image forces are used to study the dislocation interaction problems, in which several new results concerning the interactions between dislocations and anisotropic elastic inclusions are provided.
REFERENCES


Table 3.1 Hoop stress along the inclusion boundary for the point force located in the matrix

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<th>k</th>
<th>$\phi$</th>
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<th>$30^\circ$</th>
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<td>$-0.92646 \times 10^{-2}$</td>
<td>$-0.71510 \times 10^{-3}$</td>
<td>$-0.25211 \times 10^{-3}$</td>
<td>$-0.31068 \times 10^{-2}$</td>
<td>$-0.13456 \times 10^{-1}$</td>
<td>0.39014 $\times 10^{-1}$</td>
<td></td>
</tr>
<tr>
<td>$10^6$</td>
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<td>$-0.92646 \times 10^{-2}$</td>
<td>$-0.71508 \times 10^{-3}$</td>
<td>$-0.25204 \times 10^{-3}$</td>
<td>$-0.31068 \times 10^{-2}$</td>
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<td>0.12823</td>
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<td>$-0.35116$</td>
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Figure 3.1 An infinite medium containing an elliptic inclusion (elastic or rigid) or hole.
Figure 3.2a $z$-plane ($b/a = 0.6, p_\alpha = 0.3 + 1.5i$)

Figure 3.2b $z_\alpha$-plane ($b/a = 0.6, p_\alpha = 0.3 + 1.5i$)
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Figure 5.3b Contour plots of the dimensionless glide force for the anisotropic material containing a rigid inclusion with $b/a = 1$