Detailed and Efficient Analysis for Multi-Layer Thick Laminates

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摘要

本研究計打算提出一分析多層疊層板的整體–局部模式，整體局部的分割是以平面為考慮的方向而非厚度。在整體的區域，整體疊層板被考慮為均質的異向性板。而局部的區域則考慮層板為許多次層板組合而成。整體場域以位移為基本函數，局部場域則以位移和層間界面應力為獨立變數。分割的介面以幾何不連續發生的地方為準，範圍則可參考層板的厚度。準確性和收斂的情形可經由局部範圍的變更來測試。曳引引和位移的連繩性則可借由變分原理中的拉氏乘子（Lagrangian multipliers）來滿足。至於整體厚板或次層板的材料常數則採用孫錦德教授等人所建議的有效彈性係數。

關鍵詞：多層疊層板、整體–局部模式、層間介面應力、異向性、次層板、變分原理、拉氏乘子
ABSTRACT

A global-local model for the analysis of multi-layered thick laminates is proposed in this research. In the formulation, the plate is divided into two regions in the sense of plane not thickness: one is global region which considers the laminated composite as a homogeneous anisotropic plate, the other is local region which is modeled as an assemblage of sublaminates. The global field is formulated by the nodal displacements, while the local field consider the displacements and interlaminar stresses as independent variables. The division is based upon the locations where geometric discontinuities occur and the region of boundary layer which is assumed to be in the order of laminate thickness. The accuracy and convergence are tested by changing the local domain from $h$ (thickness) to $2h$, $3h$, $\cdots$ $nh$. The continuity of tractions and displacements in each sublamine is ensured by the use of Lagrangian multipliers in variational principle. The three dimensional effective elastic constants for thick laminates suggested by Sun and Li (1988) are used in global domain and every sublaminates in local domain.

Keywords: Multi-Layered laminates, Global-local model, Interlaminar stresses, Anisotropic, Sublaminates, Variational principle, Lagrangian multipliers, Multifield finite element, Singlefield finite element, Boundary element
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CHAPTER 1
INTRODUCTION

In practical applications, numerous layers may be present (use of 100 layers in aircraft structures is not unusual), but very few models have the ability of providing precise resolution of local stress fields in the vicinity of stress raisers. Classical laminated plate theory (Jones, 1975) which provides an efficient way to analyze thin composite laminates, is valid only in global sense. For moderately thick laminates, high order plate theory (Christensen, 1979) is often employed to improve accuracy. However, the two-dimensional nature of the above two theories restricts the detailed local analysis in the region near free edges, delaminations, matrix cracks, etc. A variety of methods, for examples, Pipes and Pagano (1970), Rybicki (1971), Wang and Crossman (1977), Pagano (1978a,b), Wang and Choi (1982a,b), Whitcomb, et.al. (1982), attempting to calculate the interlaminar stresses at local fields can only deal with laminates that have at most ten plies, due to the requirements of large amount computer storage and computer time. There are a few methods which are capable of analyzing multi-layered thick laminates, for example, a global-local model developed by Pagano and Soni (1983), an efficient method provided by Kassapoglou and Lagace (1986) and sublaminates modeling by Chatterjee and Ramnath (1988). In (Pagano and Soni, 1983), a laminate is divided into two parts through the thickness: local and global. The former contains a region of interest in which each layer is represented as a homogeneous anisotropic continuum and a thickness distribution of stress components within each layer is assumed. The remainder of the domain is represented by effective properties (global) and a displacement field is assumed. The efficient method (Kassapoglou and Lagace, 1986) is based upon an assumed stress function which satisfies overall force equilibrium, traction boundary and continuity conditions. However, interface displacement continuity is
not satisfied in this method. In (Chatterjee and Ramnath, 1988), a laminated composite is modeled as an assemblage of sublaminates and each sublamine is based upon the assumed linear variation of displacement and transverse shear stresses, and constant transverse normal stress through the thickness. A mixed variational principle is then applied to formulate a high order plate theory for each sublamine.

The attention of the above three methods is focused on the quasi-3D problems such as a laminate subjected to a constant axial strain. Direct extension of these methods to general 3D problems will result in some reductions in computer time and storage as compared to standard finite element methods. However, with the knowledge that the boundary layer (the range of free edge effect) is in the order of laminate thickness (Zhang and Ueng, 1988; Ye and Yang, 1988), the global-local concept can be extended from the thickness direction to plane and the computer time and storage will then be reduced substantially.

In this report, the plate is divided into two regions in the sense of plane not thickness: one is global region which considers the laminated composite as a homogeneous anisotropic plate, the other is local region which is modeled as an assemblage of sublaminates. The division is based upon the locations where geometric discontinuities, such as free edge, through-thickness cracks, delaminations, matrix cracks, etc. occur and the region of boundary layer which is assumed to be in the order of laminate thickness. The accuracy and convergence can be tested by changing the local domain from $h$ (thickness) to $2h$, $3h$, $\cdots$, $nh$. The continuity of tractions and displacements in each sublamine is ensured by the use of Lagrangian multipliers in variational principle. The three dimensional effective elastic constants for thick laminates suggested by Sun and Li (1988) are used in global domain and every sublaminates in local domain.

The implementation of global region is done by the developement of a singlefield finite
element model which can be used to analyze the general two-dimensional plane problems, and a special boundary element which can be used effectively in the analysis of global region for the laminates containing holes. For the analysis of local region, two different finite element formulations are described. One is three-dimensional multifield finite element model, the other is two-dimensional singlefield finite element model. The former is designed based upon two different kinds of functions, displacements and transverse stresses. The latter is just a displacement-based finite element. Although the singlefield finite element can be used effectively to find the solution for its two-dimensional nature, it provides the solutions in the way of ‘section by section’, i.e., it cannot provide all the field solution simultaneously. Moreover, the successful of the seperation of a 3D problem into two 2D problems (one is global region, the other is local region) is under the assumption that the thickness of composite is thinner relative to its other dimensions. No matter what kind of element has been implemented for the local region, the global-local concept is the main concern in our research. The choice of element then depends upon the type of problems considered and how accuracy is required. The overall concept of the present global-local model is finally shown by two special examples, which are done by the combination of the special boundary element for the global region and the singlefield finite element for the local region.
CHAPTER 2
GLOBAL REGION

Due to the complex stress state near the edge of composite laminates, the three
dimensional finite element analysis is usually applied even it is time consuming. In the
region far away from the edges (i.e., the global region called in this report), the classical
lamination theory provides an efficient way to analyze thin composite laminates, in which
the whole laminates are represented by a homogeneous anisotropic plate through the use
of extensional $A_{ij}$, coupling $B_{ij}$ and bending stiffness $D_{ij}$. For thick laminates, similar
concept can be employed except the three dimensional effective elastic constants should
be introduced (Sun and Li, 1988), which are derived based upon the consideration of
stress and displacement continuity conditions at the interfaces of the laminates. With this
concept in mind, the global region may be modelled as a two-dimensional homogeneous
anisotropic plate, in which the transverse components of displacements and stresses may
exist and are functions of $x$ and $y$ only. In order to establish a general purpose numerical
scheme, a displacement-based two-dimensional finite element model will be described in
section 2.1. To show the global-local concept in more complicated conditions, a special
boundary element for two-dimensional hole problems is described in section 2.2, which can
be used effectively in the analysis of the global region in laminates containing holes.

2.1 Singlefield Finite Element Models (Displacement-Based)

General Formulation

Consider the entire laminated composite as a homogeneous anisotropic plate, the total
potential energy used in the variational principle can be written as

$$\pi = \pi_G(u_i) = \int_V \left[ W(\varepsilon_{ij}) - f_i u_i \right] dV - \int_{S_e} \mathbf{T}_e^* u_i dS,$$  \hspace{1cm} (2.1)
where repeated indices imply summation. $W(\varepsilon_{ij})$ denotes the strain energy in terms of the strain components $\varepsilon_{ij}$ which are related to the displacements $u_i$ by the strain-displacement relations. $S_\sigma$ denotes part of the boundary surface where tractions $T_i^*$ are prescribed. $f_i$ is body force. In the derivation of $W(\varepsilon_{ij})$, the stress-strain law and strain-displacement relation are used. Hence,

\[ W(\varepsilon_{ij}) = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}, \quad \sigma_{ij} = C_{ijkl} \varepsilon_{kl} \]

\[ = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl}, \quad \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \]

\[ = \frac{1}{2} C_{ijkl} u_{i,j} u_{k,i}, \]

\[ = W(u_i), \]

where a comma stands for differentiation. $C_{ijkl}$ are the tensor notation of elastic constants which relate the stresses $\sigma_{ij}$ and strains $\varepsilon_{ij}$ linearly. Instead of using the conventional effective moduli (Jones, 1975) for thin composite laminates, the three dimensional effective elastic constants for thick laminates suggested by Sun and Li (1988), which are based upon the consideration of stress and displacement continuity conditions at the interfaces of the laminae, will be used for the stress-strain relations. They are

\[ \overline{C} = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\
C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\
C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\
0 & 0 & 0 & C_{44} & C_{45} & 0 \\
0 & 0 & 0 & C_{45} & C_{55} & 0 \\
C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66}
\end{bmatrix}. \]

\[ \overline{C}_{11} = \sum_{k=1}^{N} \nu_k C_{11}^{(k)} + \sum_{k=2}^{N} (C_{13}^{(k)} - \overline{C}_{13}) \nu_k (C_{13}^{(1)} - C_{13}^{(k)}) / C_{33}^{(k)}, \]

\[ \overline{C}_{12} = \sum_{k=1}^{N} \nu_k C_{12}^{(k)} + \sum_{k=2}^{N} (C_{13}^{(k)} - \overline{C}_{13}) \nu_k (C_{23}^{(1)} - C_{23}^{(k)}) / C_{33}^{(k)}, \]

\[ \overline{C}_{13} = \sum_{k=1}^{N} \nu_k C_{13}^{(k)} + \sum_{k=2}^{N} (C_{33}^{(k)} - \overline{C}_{33}) \nu_k (C_{13}^{(1)} - C_{13}^{(k)}) / C_{33}^{(k)}, \]
\[ \overline{C}_{22} = \sum_{k=1}^{N} \nu_k C_{22}^{(k)} + \sum_{k=2}^{N} (C_{23}^{(k)} - \overline{C}_{23}) \nu_k (C_{23}^{(1)} - C_{23}^{(k)}) / C_{33}^{(k)}, \]

\[ \overline{C}_{23} = \sum_{k=1}^{N} \nu_k C_{23}^{(k)} + \sum_{k=2}^{N} (C_{33}^{(k)} - \overline{C}_{33}) \nu_k (C_{23}^{(1)} - C_{23}^{(k)}) / C_{33}^{(k)}, \]

\[ \overline{C}_{33} = 1 / \left( \sum_{k=1}^{N} \nu_k / C_{33}^{(k)} \right), \]

\[ \overline{C}_{i6} = \sum_{k=1}^{N} \nu_k C_{i6}^{(k)} + \sum_{k=2}^{N} (C_{i3}^{(k)} - \overline{C}_{i3}) \nu_k (C_{i6}^{(1)} - C_{i6}^{(k)}) / C_{33}^{(k)}, \quad i = 1, 2, 3, \]

\[ \overline{C}_{66} = \sum_{k=1}^{N} \nu_k C_{66}^{(k)} + \sum_{k=2}^{N} (C_{36}^{(k)} - \overline{C}_{36}) \nu_k (C_{66}^{(1)} - C_{66}^{(k)}) / C_{33}^{(k)}, \]

\[ \overline{C}_{ij} = \left( \sum_{k=1}^{N} \nu_k C_{ij}^{(k)} / \Delta_k \right) / \Delta, \quad ij = 44, 45, 55, \]

\[ \Delta = \left( \sum_{k=1}^{N} \nu_k C_{44}^{(k)} / \Delta_k \right) \left( \sum_{k=1}^{N} \nu_k C_{55}^{(k)} / \Delta_k \right) - \left( \sum_{k=1}^{N} \nu_k C_{45}^{(k)} / \Delta_k \right)^2, \]

in which \( C_{ij}^{(k)} \) are the contracted notation of the elastic constants for the \( k \)th lamina, and

\[ \delta \pi_G = 0, \]

Combination of equations (2.1) to (2.3) shows that the functional (2.1) is expressed only by a single function \( u_i \) which is function of \( x \) and \( y \) only. We now seek the condition that the potential energy is stationary, i.e.,

\[ \delta \pi_G = 0, \]

where \( \delta \pi_G \) is obtained by the performance of calculus of variation, which leads to

\[ \delta \pi_G = \int_V [C_{ijkl} u_{k,j} \delta u_{i,j} - f_i \delta u_i] dV - \int_{S_0} T_i^* \delta u_i dS. \]

Since

\[ C_{ijkl} u_{k,j} \delta u_{i,j} = (C_{ijkl} u_{k,j} \delta u_i)_{,j} - C_{ijkl} u_{k,j} \delta u_i, \]
by the use of divergence theorem and the relation that $C_{ijkl}u_{k,s}n_j = \sigma_{ij}n_j = T_i$, equation (2.4a) becomes

$$\delta \pi_G = - \int_V (C_{ijkl}u_{k,s} + f_i) \delta u_i dV + \int_{S_e} (T_i - T_i^*) \delta u_i dS.$$  \hspace{1cm} (2.4b)

With the above results and the arbitrariness of $\delta u_i$, the Euler equation and natural boundary conditions for the present model are given by

$$C_{ijkl}u_{k,s} + f_i = 0 \quad \text{in } V,$$

$$T_i = T_i^* \quad \text{on } S_e.$$  \hspace{1cm} (2.5)

The first equation of (2.5) is the equilibrium equation in terms of displacements, while the second one is the specified boundary conditions. In finite element formulation, the displacements at any point inside the element are expressed in terms of the nodal displacements $q_{ai}$ through a set of shape functions $\mathbf{N}_a$. Hence, the only unknown function $u_i$ can be assumed as $u_i = \mathbf{N}_a q_{ai}$. The variational principle is now performed with respect to the nodal displacements $q_{ai}$, which may result in an equilibrium equation similar to (2.5)1 with unknown $q_{ai}$ instead of $u_i$. However, the tensor notation is inconvenient in numerical calculation. Like the usual outlook of finite element formulation, a matrix form expression is now described in the following.

**Numerical Procedure**

The total potential energy of the elastic body is the sum of the energy contributions of the individual elements. Thus

$$\pi = \sum_e \pi_e,$$  \hspace{1cm} (2.6a)

where $\pi_e$ represents the potential energy of element $e$ which can be written as

$$\pi_e = \frac{1}{2} \int_{V_e} \sigma^T : \dot{\varepsilon} dV - \int_{V_e} u^T \dot{f} dV - \int_{S_e} u^T \mathbf{T} : \dot{t}^* dS.$$  \hspace{1cm} (2.6b)
\( \mathbf{f} \) and \( t^* \) denote the body forces and the prescribed surface tractions, respectively. \( V_e \) is the element volume and \( S_e \) is the loaded element surface area. Assume the displacements \( \mathbf{u} \) to be expressed in terms of the nodal displacements \( \mathbf{q}_e \) through a set of shape functions \( \mathbf{N}_i \) as

\[
\mathbf{u} = \mathbf{N}_i \mathbf{q}_e. \tag{2.7a}
\]

If the eight-node isoparametric element is chosen, we have

\[
\mathbf{N} = [\mathbf{N}_1 \mathbf{N}_2 \ldots \mathbf{N}_8], \quad \mathbf{q}_e = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_8 \end{bmatrix}, \tag{2.7b}
\]

where

\[
\mathbf{N}_i = \mathbf{N}_i I_{3 \times 3}, \quad \mathbf{u}_i = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad i = 1, 2, \ldots, 8; \tag{2.7c}
\]

and the shape functions \( \mathbf{N}_i \) of eight-node isoparametric element are

\[
\begin{align*}
\mathbf{N}_1 &= \frac{1}{4} (1 + r)(1 + s) - \frac{1}{4} (1 - r^2)(1 + s) - \frac{1}{4} (1 - s^2)(1 + r), \\
\mathbf{N}_2 &= \frac{1}{4} (1 - r)(1 + s) - \frac{1}{4} (1 - r^2)(1 + s) - \frac{1}{4} (1 - s^2)(1 - r), \\
\mathbf{N}_3 &= \frac{1}{4} (1 - r)(1 - s) - \frac{1}{4} (1 - r^2)(1 - s) - \frac{1}{4} (1 - s^2)(1 - r), \\
\mathbf{N}_4 &= \frac{1}{4} (1 + r)(1 - s) - \frac{1}{4} (1 - r^2)(1 - s) - \frac{1}{4} (1 - s^2)(1 + r), \\
\mathbf{N}_5 &= \frac{1}{2} (1 - r^2)(1 + s), \\
\mathbf{N}_6 &= \frac{1}{2} (1 - s^2)(1 - r), \\
\mathbf{N}_7 &= \frac{1}{2} (1 - r^2)(1 - s), \\
\mathbf{N}_8 &= \frac{1}{2} (1 - s^2)(1 + r), \tag{2.7d}
\end{align*}
\]
in which \((r, s)\) are the local curvilinear coordinates ranging from -1 to 1, and the nodal numbers 1 to 8 are arranged in a counterclockwise sequence starting from the corner node of (1,1) as shown in Figure 1.

By the strain-displacement relation given in (2.2), the strains within the element can also be expressed in terms of the element nodal displacements as

\[
\hat{\varepsilon} = \hat{\nabla} \hat{q}_e, \quad (2.8a)
\]

where

\[
\hat{\nabla} = [\hat{R}_1 \hat{R}_2 \ldots \hat{R}_8], \quad \hat{q}_e = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}, \quad (2.8b)
\]

and

\[
\hat{R}_i = \begin{bmatrix}
\frac{\partial R_i}{\partial x} & 0 & 0 \\
0 & \frac{\partial R_i}{\partial y} & 0 \\
0 & 0 & \frac{\partial R_i}{\partial z} \\
0 & 0 & \frac{\partial R_i}{\partial x} & 0 \\
\end{bmatrix}. \quad (2.8c)
\]

The stresses are related to the strains by (2.2) as

\[
\sigma = \overline{C} \hat{\varepsilon} = \overline{\overline{C}} \hat{\nabla} \hat{q}_e, \quad (2.9a)
\]

where

\[
\sigma = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \end{Bmatrix}, \quad (2.9b)
\]

and \(\overline{C}\) has been given in (2.3) for thick composite laminates. If the body forces are neglected, substituting (2.7)-(2.9) into (2.6) leads to

\[
\pi_e = \frac{1}{2} \int_{V_e} \hat{q}_e^T \hat{\nabla}^T \overline{\overline{C}} \hat{\nabla} \hat{q}_e dV_e - \int_{S_e} \hat{q}_e^T \hat{\nabla}^T \hat{\tau}^* dS_e. \quad (2.10)
\]
Performance of the minimization for element $e$ with respect to the nodal displacement $\mathbf{q}_e$ for the element results in

$$\frac{\partial \pi_e}{\partial \mathbf{q}_e} = \{ \int_{V_e} \mathbf{N}_e^T \mathbf{C}_e \mathbf{N}_e dV_e \} \mathbf{q}_e - \int_{S_e} \mathbf{N}_e^T \mathbf{t}^* dS_e = 0,$$  

(2.11a)

or,

$$K_e \mathbf{q}_e = \mathbf{Q}_e,$$  

(2.12a)

where

$$K_e = \int_{V_e} \mathbf{N}_e^T \mathbf{C}_e \mathbf{N}_e dV_e,$$  

(2.12b)

$$\mathbf{Q}_e = \int_{S_e} \mathbf{N}_e^T \mathbf{t}^* dS_e.$$  

$K_e$ is termed the element stiffness matrix and $\mathbf{Q}_e$ is the equivalent nodal force. The summation of the terms in (2.12) over all the elements results in a system of equilibrium equations for the complete continuum. These equations are then solved by any standard technique to yield the nodal displacements. The stresses and strains within each element can then be calculated from the nodal displacements using (2.8) and (2.9).

### 2.2 A Special Boundary Element for Two-Dimensional Hole Problems

In this section, a special boundary element is introduced to solve the plane problems of composite laminates containing an elliptic hole. The fundamental solution needed for the boundary element method is chosen to be the one which satisfies the traction-free hole boundary conditions a *priori*. Therefore, the discretization around the hole boundary is avoided, which results in a saving of computer time and storage. Moreover, discretization with relatively coarse meshes can achieve high accuracy. As stated in the last section, the results obtained by this boundary element analyses are valid only in the global region of the laminates, i.e., the region away from the edges by a distance of the order of magnitude $h$. 

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Boundary integral equation

If body forces are omitted, the boundary integral equation is written as (Brebbia, et.al., 1984)

$$\lambda_{ij}(\mathbf{\hat{z}})u_{j}(\mathbf{\hat{z}}) + \int_{B+L} t_{ij}^{s}(\mathbf{\hat{z}}, \mathbf{\bar{z}})u_{j}(\mathbf{\bar{z}})d\Gamma(\mathbf{\bar{z}}) = \int_{B+L} u_{ij}^{s}(\mathbf{\hat{z}}, \mathbf{\bar{z}})t_{j}(\mathbf{\bar{z}})d\Gamma(\mathbf{\bar{z}}),$$  \hspace{1cm} (2.13)

where $u_{ij}^{s}(\mathbf{\hat{z}}, \mathbf{\bar{z}})$ and $t_{ij}^{s}(\mathbf{\hat{z}}, \mathbf{\bar{z}})$ are, respectively, the displacements and tractions in the $j$ direction at point $\mathbf{\bar{z}} = (x_1, x_2)$ corresponding to a unit point force acting in the $i$ direction applied at point $\mathbf{\hat{z}} = (\hat{x}_1, \hat{x}_2)$ and $B, L$ denote the contour of the outer and hole boundary. $u_{j}(\mathbf{\bar{z}})$ and $t_{j}(\mathbf{\bar{z}})$ are the displacements and surface tractions along the boundaries. $\lambda_{ij}(\mathbf{\hat{z}})$ is a coefficient to be determined by the boundary geometry. For a smooth boundary $\lambda_{ij} = \frac{1}{2}\delta_{ij}$, in which $\delta_{ij}$ is the Kronecker delta. In general, the value of $\lambda_{ij}(\mathbf{\hat{z}})$ can be estimated by considering the rigid body translation.

For the traction-free hole problem, we choose $u_{ij}^{s}$ and $t_{ij}^{s}$ in such a way that $t_{ij}^{s} = 0$ on the hole boundary $L$. Then, by applying the boundary condition $t_{j} = 0$ on $L$, we find that

$$\int_{L} t_{ij}^{s}(\mathbf{\hat{z}}, \mathbf{\bar{z}})u_{j}(\mathbf{\bar{z}})d\Gamma(\mathbf{\bar{z}}) = \int_{L} u_{ij}^{s}(\mathbf{\hat{z}}, \mathbf{\bar{z}})t_{j}(\mathbf{\bar{z}})d\Gamma(\mathbf{\bar{z}}) = 0,$$  \hspace{1cm} (2.14)

which means that the integrals along the hole boundary vanish and the discretization around the hole can be avoided. An appropriate solution for $u_{ij}^{s}$ and $t_{ij}^{s}$ can be found by using the Green’s function of two-dimensional anisotropic plates containing an elliptic hole (Hwu and Yen, 1991), which is

$$u = \frac{1}{\pi} \text{Im} \{ \mathbf{A} \ll \ln(\zeta_{\alpha} - \zeta_{\alpha}) \gg \mathbf{A}^{T} \} \mathbf{l}_{k} + \frac{1}{\pi} \sum_{k=1}^{3} \text{Im} \{ \mathbf{B} \ll \ln(\zeta_{k}^{-1} - \zeta_{k}) \gg \mathbf{B}^{-1} \mathbf{B}_{l} \mathbf{A}^{T} \} \mathbf{l}_{k},$$

$$\phi = \frac{1}{\pi} \text{Im} \{ \mathbf{B} \ll \ln(\zeta_{\alpha} - \zeta_{\alpha}) \gg \mathbf{A}^{T} \} \mathbf{l}_{k} + \frac{1}{\pi} \sum_{k=1}^{3} \text{Im} \{ \mathbf{B} \ll \ln(\zeta_{k}^{-1} - \zeta_{k}) \gg \mathbf{B}^{-1} \mathbf{B}_{l} \mathbf{A}^{T} \} \mathbf{l}_{k},$$  \hspace{1cm} (2.15a)
where
\[
\zeta_\alpha = \frac{z_\alpha + \sqrt{z_\alpha^2 - a^2 - p_\alpha^2 b^2}}{a - ip_\alpha b}, \quad \hat{\zeta}_\alpha = \frac{\hat{z}_\alpha + \sqrt{\hat{z}_\alpha^2 - a^2 - p_\alpha^2 b^2}}{a - ip_\alpha b},
\]
\[
z_\alpha = x_1 + p_\alpha x_2, \quad \hat{z}_\alpha = \hat{x}_1 + p_\alpha \hat{x}_2,
\]
\[
I_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.
\]

\[ (2.15b) \]

In the above, \( u \) and \( \phi \) denote, respectively, the vectors of displacements \( (u_1, u_2, u_3) \), and stress functions \( (\phi_1, \phi_2, \phi_3) \); \( \hat{f} \) is the unit point load applied at \( (\hat{x}_1, \hat{x}_2) \); \( p_\alpha, \alpha = 1, 2, 3 \) denote the material eigenvalues which are complex numbers with positive imaginary part; \( \tilde{A}, \tilde{B} \) are the \( 3 \times 3 \) complex square matrices composed of the elasticity constants \( C_{ijks} \); the over bar stands for the complex conjugate; the superscript \( T \) denotes the transpose of matrices; \( \text{Im} \) denotes the imaginary part and the angular bracket stands for the diagonal matrix, i.e., \( \langle f_\alpha \rangle = \text{diag}\{f_1, f_2, f_3\} \).

It is known that the stress function \( \phi \) is related to the surface traction \( t \) by \( t = \partial \phi / \partial s \) where \( s \) is the arc length measured along the curved boundary. By using this relation and applying the Green’s function given in (2.15), \( U^*_\sim = [u^*_i] \), \( T^*_\sim = [t^*_i] \) are obtained as
\[
U^*_\sim = \frac{1}{\pi} \text{Im} \left\{ A \ll \log(\zeta_\alpha - \hat{\zeta}_k) \gg A^T \right\}
+ \frac{1}{\pi} \sum_{k=1}^3 \text{Im} \left\{ B \ll \frac{2\zeta_\alpha^2(s_1 + p_\alpha s_2)}{(\zeta_\alpha - \hat{\zeta}_k)(a - ip_\alpha b)\zeta_\alpha^2 - (a + ip_\alpha b)} \gg A^T \right\}
\]
\[
T^*_\sim = \frac{1}{\pi} \text{Im} \left\{ B \ll \frac{2\zeta_\alpha(s_1 + p_\alpha s_2)}{(1 - \zeta_\alpha \hat{\zeta}_k)(a - ip_\alpha b)\zeta_\alpha^2 - (a + ip_\alpha b)} \gg B^1 \ll \tilde{I}_k \tilde{A}^T \right\}
\]
\[
(2.16)
\]

where \( s_1 = \frac{\partial \zeta_1}{\partial s} \), \( s_2 = \frac{\partial \zeta_2}{\partial s} \).

If \( \hat{\zeta} \) is an internal point, \( \lambda_{ij} \) in (2.13) becomes \( \delta_{ij} \). Equation (2.13) is a continuous representation of displacement at any point \( \hat{\zeta} \) inside the body. The internal stresses at point \( \hat{\zeta} \) can be found by differentiating (2.13) with respect to \( \hat{\zeta} \) and using the strain-
displacement relation $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ and stress-strain law $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$. The results are

$$
\begin{align*}
\sigma_1(\hat{\mathbf{x}}) &= \int_B \left( C_{11} \frac{\partial U^*(\hat{\mathbf{x}}, x)}{\partial \hat{x}_1} + C_{12} \frac{\partial U^*(\hat{\mathbf{x}}, x)}{\partial \hat{x}_2} \right) \mathbf{t}(x) \, d\Gamma(x), \\
\sigma_2(\hat{\mathbf{x}}) &= \int_B \left( C_{12} \frac{\partial T^*(\hat{\mathbf{x}}, x)}{\partial \hat{x}_1} + C_{22} \frac{\partial T^*(\hat{\mathbf{x}}, x)}{\partial \hat{x}_2} \right) \mathbf{t}(x) \, d\Gamma(x),
\end{align*}
$$

(2.17a)

$$
\begin{align*}
\sigma_1(\hat{\mathbf{x}}) &= \int_B \left( C_{11} \frac{\partial U^*(\hat{\mathbf{x}}, x)}{\partial \hat{x}_1} + C_{12} \frac{\partial U^*(\hat{\mathbf{x}}, x)}{\partial \hat{x}_2} \right) \mathbf{u}(x) \, d\Gamma(x), \\
\sigma_2(\hat{\mathbf{x}}) &= \int_B \left( C_{12} \frac{\partial T^*(\hat{\mathbf{x}}, x)}{\partial \hat{x}_1} + C_{22} \frac{\partial T^*(\hat{\mathbf{x}}, x)}{\partial \hat{x}_2} \right) \mathbf{u}(x) \, d\Gamma(x),
\end{align*}
$$

(2.17b)

where $C_{11}$, $C_{12}$ and $C_{22}$ are $3 \times 3$ real matrices given by

$$
(C_{11})_{ik} = C_{i1k1}, \quad (C_{12})_{ik} = C_{i1k2}, \quad (C_{22})_{ik} = C_{i2k2},
$$

(2.17c)

and

$$
\begin{align*}
\hat{\mathbf{t}} &= \left\{ \frac{\sigma_{11}}{\sigma_{13}} \right\}, \quad \hat{\mathbf{u}} = \left\{ \frac{\sigma_{21}}{\sigma_{23}} \right\}, \\
\overline{\mathbf{t}} &= \left\{ t_1, t_2, t_3 \right\}, \quad \overline{\mathbf{u}} = \left\{ u_1, u_2, u_3 \right\}.
\end{align*}
$$

(2.17d)

Numerical procedure

The boundary integral equation (2.13) must, in general, be solved numerically. Here linear elements are used, so that the values of $\mathbf{u}$ and $\mathbf{t}$ over each element along the boundary can be expressed in terms of nodal displacements $u_m$ and traction $t_m$ as

$$
\mathbf{u} = \overline{\mathbf{u}} u_m, \quad \mathbf{t} = \overline{\mathbf{t}} t_m,
$$

(2.18a)

where

$$
\begin{align*}
\overline{\mathbf{u}} &= \left\{ \bar{u}_1, \bar{u}_2 \right\}_{3 \times 6}, \quad \overline{\mathbf{t}} = \left\{ \bar{t}_1 \right\}_{6}, \\
\underline{\mathbf{u}}_i &= \left\{ u_1, u_2, u_3 \right\}_i, \quad \underline{\mathbf{t}}_i = \left\{ t_1, t_2, t_3 \right\}_i, \quad i = 1, 2,
\end{align*}
$$

(2.18b)
and the shape functions $\varpi_i$ are
\[
\varpi_1 = \frac{1}{2}(1 - \xi), \quad \varpi_2 = \frac{1}{2}(1 + \xi).
\] (2.18c)

Here, $\xi$ is the dimensionless coordinate defined by $\xi = 2s/\ell$ where $\ell$ is the length of the element considered and $s$ is the coordinate lying along the linear element and directing from the first node to the second node of element $m$.

If the boundary $B$ is discretized into $M$ segments with $N$ nodes, substitution (2.18) into (2.13) with (2.14) yields
\[
\hat{\Lambda}_i \underline{u}(\hat{\xi}) + \sum_{m=1}^{M} \left( \int_{\Gamma_m} T^* \varpi d\Gamma_m \right) \underline{u}_m = \sum_{i=m}^{M} \left( \int_{\Gamma_m} U^* \varpi d\Gamma_m \right) \underline{t}_m,
\] (2.19)

where $\Gamma_m$ denotes the $m$th segment of the discretized boundary. $\int_{\Gamma_m} T^* \varpi d\Gamma_m$ and $\int_{\Gamma_m} U^* \varpi d\Gamma_m$ are the matrices of influence coefficients defining the interaction between the point $\hat{\xi}$ under consideration and the particular node (1 and 2) on element $m$. $\hat{\Lambda}_i$ is the matrix of coefficient $\lambda_{ij}$ at point $\hat{\xi}$. To write the equation corresponding to node $i$ in discrete form (consider $\hat{\xi}$ to be node $i$), we need to add the contribution from two adjoining elements, $m$ and $m - 1$, into one term, defining the nodal coefficient. This will give the following equation
\[
\hat{\Lambda}_i \underline{u}_i + [\hat{H}_{i1} \hat{H}_{i2} \ldots \hat{H}_{iN}] \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} = [G_{i1} G_{i2} \ldots G_{iN}] \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_N \end{bmatrix},
\] (2.20)

where each $\hat{H}_{ij}$ term is equal to $\int_{\Gamma_{m-1}} T^* \varpi_2 d\Gamma_{m-1}$ of element $m - 1$ plus $\int_{\Gamma_m} T^* \varpi_1 d\Gamma_m$ of element $m$ for an anticlockwise numbering system. The same applies for $G_{ij}$. Hence, equation (2.20) represents the assembled equation for node $i$ and can be written as
\[
\sum_{j=1}^{N} \hat{H}_{ij} u_j = \sum_{j=1}^{N} G_{ij} t_j,
\] (2.21a)
where
\[ \tilde{H}_{ij} = \hat{H}_{ij} \quad \text{for} \quad i \neq j, \]
\[ \tilde{H}_{ij} = \hat{H}_{ij} + \Lambda_i \quad \text{for} \quad i = j. \]  

(2.21b)

When all the nodes are taken into consideration, equation (2.21) produces a \(3N \times 3N\) system of equations which can be represented in matrix form as

\[
\begin{bmatrix}
\tilde{H}_{11} & \tilde{H}_{12} & \cdots & \tilde{H}_{1N} \\
\tilde{H}_{21} & \tilde{H}_{22} & \cdots & \tilde{H}_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{H}_{i1} & \tilde{H}_{i2} & \cdots & \tilde{H}_{iN} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{H}_{N1} & \tilde{H}_{N2} & \cdots & \tilde{H}_{NN}
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_1 \\
\tilde{u}_2 \\
\vdots \\
\tilde{u}_i \\
\vdots \\
\tilde{u}_N
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{G}_{11} & \tilde{G}_{12} & \cdots & \tilde{G}_{1N} \\
\tilde{G}_{21} & \tilde{G}_{22} & \cdots & \tilde{G}_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{G}_{i1} & \tilde{G}_{i2} & \cdots & \tilde{G}_{iN} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{G}_{N1} & \tilde{G}_{N2} & \cdots & \tilde{G}_{NN}
\end{bmatrix}
\begin{bmatrix}
\tilde{t}_1 \\
\tilde{t}_2 \\
\vdots \\
\tilde{t}_i \\
\vdots \\
\tilde{t}_N
\end{bmatrix}.
\]

(2.22)

By applying the boundary condition where either \(u_i\) or \(t_i\) at each node is prescribed, the system of equations (2.22) can be reordered in such a way that the final system of equations can be expressed as \(Kq = Q\) where \(K\) is a fully populated matrix, \(q\) is a vector containing all the boundary unknowns and \(Q\) is vector containing all the prescribed values given on the boundary. Once all the values of tractions and displacements on the boundary are determined, the values of stresses and displacements at any interior point can be calculated from (2.13) and (2.17).

Note that the coefficients \(\Lambda_i\) together with the corresponding principal value of \(\tilde{H}_{ij}\) are indirectly computed by consideration of rigid-body movements in the body i.e., by letting \(u_j = 1\) (and hence \(t_j = 0\)) in equation (2.22). The integrals including the one with a singular term such as \(\log(\zeta_\alpha - \check{\zeta}_\alpha)\) and \(1/(\zeta_\alpha - \check{\zeta}_\alpha)\) in (2.16) are evaluated by using Gauss quadrature rules. The required Gauss points for integration depend on the distance between the point under consideration and the midpoint of each element because the smaller the distance the larger the variation of tractions and displacements. For the point located on the boundary, 16 Gauss points are used to have a convergence solution for the singular integrals.
CHAPTER 3
LOCAL REGION

When a thick composite laminate is analyzed by the element developed in the last chapter for the global region, the traction boundary conditions are satisfied only in an integrated sense. When the stresses in each ply are calculated from the ply stress-strain law, it will be found that the boundary conditions in each ply along the free edges may not be satisfied. To satisfy the traction-free boundary conditions along the free edges for each ply, an additional stresses should be prescribed in the local region along the free edges. That is,

$$\sigma_{yz} = -\sigma_{y'z'}, \quad \sigma_{yy} = -\sigma_{y'y'}, \quad \sigma_{yz} = -\sigma_{y'z'}$$

(3.1)

where $(x, y, z)$ is a cartesian coordinate system chosen for the local region. The $x$-axis and $y$-axis lying along the midplane of the laminates are, respectively, tangent and normal to the edge boundary, while $z$-axis is perpendicular to the midplane. $\sigma_{ij}^{(G)}$ are the stresses in each ply along the free edges calculated in the global region. Since the global region already provides good approximation for the region far away from the edges, the additional stresses $\sigma_{ij}$ determined from the local region should satisfy

$$\lim_{y \to \infty} \sigma_{ij} = 0.$$  

(3.2)

Knowing that the boundary layer effect, which the local region concerns, penetrates from the edges into the plate only a distance of the order of magnitude in thickness $h$. The accuracy and convergence will be tested by changing the local region in $y$-direction from $h$ to $2h$, $3h$, ..., $nh$. To expedite the analysis, the whole laminate with $N$ laminae in the local region can be further subdivided into $n$ sublaminates where the $i$th sublamine $(i = 1, 2, 3, ..., n)$ contains $P_i$ laminae which are assumed to be perfectly bonded to each other. The effective moduli of each sublaminate can also be calculated by equation (2.3).
A universal and widely accepted method to deal with the problems restricted by the above conditions is the finite element method. In the following, two different finite element formulations will be described. One is multifield finite element models, the other is singlefield finite element model. The former is designed based upon two different kinds of functions, displacements and transverse stresses, in order to obtain a good result satisfying the continuity conditions of interlaminar stresses and displacements. The element used in this formulation is in general a hybrid 3D element, which provides all the numerical field solutions simultaneously. The latter will be designed based upon only one kind of functions, displacements, which are functions of \( y \) and \( z \) only. The formulation will be the same as that described in Section 2.1, except that the plane considered now is \( yz \) plane not \( xy \) plane. Since it keeps two-dimensional nature of the finite element, it will save a vast of computer time and storage. Moreover, because the element is displacement-based, the continuity of displacements will be satisfied exactly, and the interlaminar stresses continuity will also be satisfied in the variational sense. However, the seperation of a 3D elasticity problem into two 2D problems is valid only for thin composite laminates. Moreover, the two-dimensional formulation cannot provide all the field solution simultaneously, it gives the solution only in the way of ‘section by section’. That is, if one is interested in a particular point along the boundary edges, one may cut a cross section containing this point, which extends in \( y \)-direction from the edge to a distance of the order of magnitude \( h \), and in \( z \)-direction from the bottom to the top of the laminates. For all other interesting points, a totally new calculation should be performed. In practical applications, one may choose any one of them based upon the need of the problems considered, and the advantages and disadvantages described previously.
3.1 Multifield Finite Element Models

General Formulation

In order to satisfy the continuity of interlaminar stresses and displacements simultaneously, the strain-displacement relations in the thickness direction will be relaxed by the use of Lagrangian multipliers. Therefore, the functional used in the local region for the mixed variational principle can be written as

$$I_L = \sum_{k=1}^{n} \int_{V_k} \left\{ \left[ W^k - f_i^k u_i^k \right] + \sigma_{ij}^k \left[ \frac{1}{2} (u_{ij}^k + u_{ji}^k) - \varepsilon_{ij}^k \right] \right\} dV_k - \int_{S_e} T_i^* u_i dS, \quad (3.3a)$$

where $i$ and $j$ vary from 1 to 3, $\gamma \delta$ varies over 13, 23, 33, 32, 31. $k$ is the index for the sublamine and $n$ is the number of sublaminates.

Substituting the strain energy of $k$th sublamine, $W^k = \frac{1}{2} \sigma_{ij}^k \varepsilon_{ij}^k$, into equation (3.3a) and defining a new energy expression $\hat{W}^k$ as

$$\hat{W}^k = \frac{1}{2} \sigma_{ij}^k \varepsilon_{ij}^k, \quad (3.3b)$$

where the subscript $\alpha \beta$ varies over 11, 12, 21, 22, we have

$$I_L = \sum_{k=1}^{n} \int_{V_k} \left[ \hat{W}^k - \sigma_{ij}^k \varepsilon_{ij}^k + 2 \sigma_{ij}^k (u_{ij}^k + u_{ji}^k) - f_i^k u_i^k \right] dV_k - \int_{S_e} T_i^* u_i dS. \quad (3.3c)$$

If the energy expression $\hat{W}^k$ which depends on the material properties of $k$th sublamine is chosen to be a function of $\varepsilon_{ij}^k \sigma_{ij}^k$ only and satisfies the constitutive laws such that

$$\frac{\partial \hat{W}^k}{\partial \varepsilon_{ij}^k} = \sigma_{ij}^k, \quad \frac{\partial \hat{W}^k}{\partial \sigma_{ij}^k} = -\varepsilon_{ij}^k, \quad (3.4)$$

the functional $I_L$ will have three independent arguments, i.e., $I_L = I_L (u_i, \varepsilon_{ij}, \sigma_{ij})$. Moreover, if the inplane strain components $\varepsilon_{ij}^k$ are related to the displacements $u_i$ by the linear strain-displacement relation such that

$$\varepsilon_{ij}^k = \frac{1}{2} (u_{ij}^k + u_{ji}^k), \quad (3.5)$$
the functional $I_L$ becomes

$$I_L = I_L(u_i, \sigma_{\gamma \delta}).$$

(3.6)

Performance of calculus of variation on (3.3c) leads to

$$\delta I_L = \sum_{k=1}^{n} \int_{V_k} \left[ \frac{\partial W_k^k}{\partial e^{\alpha \beta}_{\alpha \beta}} \delta e^{\alpha \beta}_{\alpha \beta} + \frac{\partial W_k^k}{\partial \sigma_{\gamma \delta}} \delta \sigma_{\gamma \delta}^k - \sigma_{\alpha \beta}^k \delta \varepsilon_{\alpha \beta}^k - \varepsilon_{\alpha \beta}^k \delta \sigma_{\alpha \beta}^k + \frac{1}{2}(u_{i,j} + u_{j,i})\delta \sigma_{ij}^k \right] dV_k + \frac{1}{2} \sigma_{ij}^k \delta u_{i,j}^k$$

$$+ f_i^k \delta u_i^k |_{dV_k} - \int_{S_{\sigma}} T_i^* \delta u_i dS.$$  

(3.7)

Substituting the following equation into (3.7)

$$\frac{1}{2} \sigma_{ij}^k \delta (u_{i,j}^k + u_{j,i}^k) = (\sigma_{ij}^k \delta u_i^k)_{,j} - \sigma_{ij,j}^k \delta u_i^k,$$

and using the divergence theorem and the relations given in (3.4) and (3.5), the stationary condition $\delta I_L = 0$ will now provide

$$\varepsilon_{\gamma \delta}^k = \frac{1}{2}(u_{i,j}^k + u_{j,i}^k), \quad \text{in } V_i,$$

$$\sigma_{ij}^k + f_i^k = 0, \quad \text{in } V_i,$$

$$\sigma_{ij} n_j = T_i^*, \quad \text{on } S_{\sigma}.$$  

(3.8)

By comparing (3.8) with (2.5), we see that the main difference between the displacement-based and present finite element model is the strain-displacement relation in transverse direction. In the displacement-based finite element model, this relation is satisfied exactly and substituted into the formulation as a given condition. While in the present model, this relation is satisfied in the variational sense and is relaxed through the use of Lagrange's multipliers. In other words, if all the strain-displacement relations are used, the present model will reduce to the usual displacement-based finite element. However, in order to overcome the discontinuity of interlaminar stresses, the present model is thought to be better than the displacement-based finite element because relaxing the strain-displacement relation in transverse direction will let the transverse stresses have more freedom to find their most
suitable values variationally. Otherwise, they are destined to be discontinuous, which is physically unreasonable for a continuum.

In order to write the functional $I_L$ in terms of $u_i$ and $\sigma_{\gamma\delta}$ explicitly, in what follows we introduce a modified contracted notation, which is different from the conventional one, as

$$
\begin{align*}
\sigma_{11} &= \sigma_1, & \sigma_{22} &= \sigma_2, & \sigma_{12} &= \sigma_3, \\
\sigma_{33} &= \sigma_4, & \sigma_{31} &= \sigma_5, & \sigma_{32} &= \sigma_6, \\
\varepsilon_{11} &= \varepsilon_1, & \varepsilon_{22} &= \varepsilon_2, & 2\varepsilon_{12} &= \varepsilon_3, \\
\varepsilon_{33} &= \varepsilon_4, & 2\varepsilon_{31} &= \varepsilon_5, & 2\varepsilon_{32} &= \varepsilon_6.
\end{align*}
\tag{3.9}
$$

The relation between stresses and strains in the $k$th sublaminate may then be expressed in terms of the modified effective stiffness matrix $C^*$ as

$$
\sigma_i = C^*_{ij}\varepsilon_j + C^*_{i\beta}\varepsilon_\beta, \tag{3.10a}
$$

$$
\sigma_\alpha = C^*_{\alpha j}\varepsilon_j + C^*_{\alpha \beta}\varepsilon_\beta, \tag{3.10b}
$$

where $i, j$ vary over 1, 2 and 3, and $\alpha, \beta$ vary over 4, 5 and 6. By solving $\varepsilon_\beta$ and $\sigma_i$ in terms of $\varepsilon_i$ and $\sigma_\alpha$, we have

$$
\varepsilon_\beta = S^*_{\alpha \beta}\sigma_\alpha - S^*_{\alpha \beta}C^*_{\alpha j}\varepsilon_j, \tag{3.10c}
$$

$$
\sigma_i = C^*_{i\beta}S^*_{\alpha \beta}\sigma_\alpha + [C^*_{ij} - C^*_{i\beta}S^*_{\alpha \beta}C^*_{\alpha j}]\varepsilon_j,
$$

where

$$
S^*_{\alpha \beta} = C^*_{\alpha \beta}^{-1}. \tag{3.10c}
$$

The energy expression $\hat{W}$ given in (3.3b) can then be expressed as

$$
\hat{W} = \frac{1}{2}[C^\prime_{ij}\varepsilon_i\varepsilon_j + 2C^\prime\prime_{ia}\varepsilon_i\sigma_\alpha - S^*_{\alpha \beta}\sigma_\alpha\sigma_\beta], \tag{3.11a}
$$

where

$$
C^\prime_{ij} = C^*_{ij} - C^*_{i\beta}S^*_{\alpha \beta}C^*_{\alpha j} = C^*_{ij} - C^\prime\prime_{ia}C^*_{\alpha j}, \tag{3.11b}
$$

$$
C^\prime\prime_{ia} = C^*_{i\beta}S^*_{\alpha \beta}.
$$
Substituting (3.11) into (3.3c) and using the modified contracted notation, the functional $I_L$ can now be expressed as

$$I_L = I_L(u_i, \sigma_\alpha) = \sum_{k=1}^{n} \int_{V_k} \left[ \frac{1}{2} (C_{ij}^{\mu k} \varepsilon_i^k \varepsilon_j^k + 2C_{i\alpha}^{\mu k} \varepsilon_i^k \sigma_\alpha^k - S_{\alpha \beta}^{* k} \sigma_\alpha^k \sigma_\beta^k) + \sigma_4^k u_{3,1}^k + \sigma_5^k (u_{3,2}^k + u_{2,3}^k) ight] dV_k - \int_{S_e} T^*_i u_i dS,$$

(3.12)

where the inplane strain $\varepsilon_i, i = 1, 2, 3$, are related to the displacements by equation (3.5).

**Numerical Procedure**

Through dividing the local region into several elements, the functional $I_L$ given in (3.12) can be considered to be the sum contributed by each individual element. Thus,

$$I_L = \sum_e I_e,$$

where, in matrix form,

$$I_e = \sum_{k=1}^{n} \int_{V_k} \left[ \frac{1}{2} \varepsilon_{p}^{kT} C^{\mu k} \varepsilon_{p}^{k} + \varepsilon_{p}^{kT} \sigma_{t}^{k} - \frac{1}{2} \sigma_{t}^{kT} S^{* k} \sigma_{t}^{k} - \sigma_{4}^{k} u_{3,1}^{k} - \sigma_{5}^{k} (u_{3,2}^{k} + u_{2,3}^{k}) \right] dV_k$$

$$+ \sigma_{t}^{kT} D^{k} \tilde{u}^{k} - \tilde{f}^{kT} \tilde{u}^{k} \right] dV_k - \int_{S_{e}} \tilde{u}^{kT} T^{* k} dS_e.$$

(3.13)

The subscripts $p$ and $t$ denote, respectively, the inplane and transverse portion of the stresses and strains. No summation is implied for the repeated index in matrix notation.

The components of each notations are

$$\varepsilon_{p}^{k} = \begin{pmatrix} \varepsilon_{11}^{k} \\ \varepsilon_{22}^{k} \\ 2\varepsilon_{12}^{k} \end{pmatrix},$$

(3.14a)

$$\sigma_{t}^{k} = \begin{pmatrix} \sigma_{4}^{k} \\ \sigma_{5}^{k} \\ \sigma_{6}^{k} \end{pmatrix},$$

(3.14b)

$$\tilde{u}^{k} = \begin{pmatrix} u_{1}^{k} \\ u_{2}^{k} \\ u_{3}^{k} \end{pmatrix},$$

$$\tilde{f}^{k} = \begin{pmatrix} f_{1}^{k} \\ f_{2}^{k} \\ f_{3}^{k} \end{pmatrix},$$

$$T^{* k} = \begin{pmatrix} T_{1}^{*} \\ T_{2}^{*} \\ T_{3}^{*} \end{pmatrix}.$$
\[ D_i = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix}. \]  

(3.14d)

\( \tilde{C}^{\nu k} \) \( \tilde{C}^{\mu k} \) and \( \tilde{S}^{\star k} \) are the matrix form of \( C_{\nu j}^{\mu} \), \( C_{\mu k}^{\nu} \) and \( S_{\alpha\beta}^{\star} \) of the kth sublamine given in equations (3.11b) and (3.10c).

To be compatible with the singlefield finite element described in the last chapter, a three-dimensional isoparametric twenty-node hexahedron element is chosen for the displacement representation. That is

\[ u^k = \mathcal{N} g^k, \]

(3.15a)

where

\[ \mathcal{N} = [\mathcal{N}_1 \mathcal{N}_2 \ldots \mathcal{N}_{20}], \quad g^k = \begin{bmatrix} u^k_1 \\ u^k_2 \\ \vdots \\ u^k_{20} \end{bmatrix}, \]

(3.15b)

and

\[ \mathcal{N}_i = \mathcal{N}_i I_{3 \times 3}, \quad u^k_i = \begin{bmatrix} u^k_1 \\ u^k_2 \\ u^k_3 \end{bmatrix}, \quad i = 1, 2, \ldots, 20. \]

(3.15c)

The shape functions \( \mathcal{N}_i \) of the twenty-node isoparametric solid element are

\( \mathcal{N}_j = \frac{1}{8} (1 + \xi_j \zeta_j)(1 + \eta_j \zeta_j)(1 + \zeta_j \zeta_j)(-2 + \xi_j \xi_j + \eta_j \eta_j + \zeta_j \zeta_j). \)

(3.15d)

(2) For nodes on the sides parallel to axis \( \xi \) \( (\xi_j = 0 \text{ and } \xi_j, \zeta_j = -1 \text{ or } 1, j=1,2,\ldots,8) \)

\[ \mathcal{N}_j = \frac{1}{4}(1 - \xi_j^2)(1 + \eta_j \zeta_j)(1 + \zeta_j \zeta_j). \]

(3.15e)

(3) For nodes on the sides parallel to axis \( \eta \) \( (\eta_j = 0 \text{ and } \eta_j, \zeta_j = -1 \text{ or } 1, j=10,12,14,16) \)

\[ \mathcal{N}_j = \frac{1}{4}(1 + \xi_j \xi_j)(1 - \eta_j^2)(1 + \zeta_j \zeta_j). \]

(3.15f)
(4) For nodes on the sides parallel to axis $\zeta$ ($\zeta_j = 0$ and $\xi_j, \eta_j = -1$ or 1, $j=9,11,13,15$)

$$\xi_j = \frac{1}{4}(1 + \xi_j \xi)(1 + \eta_j \eta)(1 - \zeta^2).$$  \hspace{1cm} (3.15g)

Note that $(\xi, \eta, \zeta)$ are the local curvilinear coordinates ranging from $-1$ to 1. By the strain-displacement relation given in (3.5), the inplane strains within the element can be expressed in terms of the element nodal displacements as

$$\tilde{\varepsilon}_p^k = \tilde{B}_p \tilde{q}_e^k,$$  \hspace{1cm} (3.16a)

where

$$\tilde{B} = [\tilde{B}_1 \tilde{B}_2 \ldots \tilde{B}_{20}],$$  \hspace{1cm} (3.16b)

and

$$\tilde{B}_i = \begin{bmatrix} \frac{\partial \tilde{N}_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial \tilde{N}_i}{\partial y} & 0 \\ \frac{\partial \tilde{N}_i}{\partial y} & \frac{\partial \tilde{N}_i}{\partial x} & 0 \end{bmatrix}.$$  \hspace{1cm} (3.16c)

The fourth term in the volume integral of equation (3.13) can also be expressed in terms of the element nodal displacements as

$$D_i \tilde{u}_e^k = \tilde{B}_i \tilde{q}_e^k,$$  \hspace{1cm} (3.17a)

where

$$\tilde{B} = [\tilde{B}_1 \tilde{B}_2 \ldots \tilde{B}_{20}],$$  \hspace{1cm} (3.17b)

and

$$\tilde{B}_i = \begin{bmatrix} 0 & 0 & \frac{\partial \tilde{N}_i}{\partial x} \\ 0 & \frac{\partial \tilde{N}_i}{\partial y} & \frac{\partial \tilde{N}_i}{\partial x} \\ \frac{\partial \tilde{N}_i}{\partial y} & 0 & 0 \end{bmatrix}.$$  \hspace{1cm} (3.17c)

The transverse stresses $\sigma_t^k$ is assumed independently as

$$\sigma_t^k = T^k \tilde{P}^k \tilde{\zeta}_e^k,$$  \hspace{1cm} (3.18)
where $\tilde{\beta}_e^k$ is a stress parameter vector, $\tilde{P}^k$ is an assumed stress function matrix, i.e., a stress pattern which is expressed in a local coordinate system $(\xi, \eta, \zeta)$, and the stress transformation matrix $\tilde{T}^k$ is used to transform the transverse stresses from the local coordinate system to the global $(x, y, z)$ system (Cook, 1981). The choice of the assumed stress function $\tilde{P}^k$ is based upon (1) the consistency with the displacement gradient, (2) interface traction continuity, (3) the traction-free condition at the top and bottom surface. A detailed description of the assumed stress function $\tilde{P}^k$ and the stress parameter vector $\tilde{\beta}_e^k$ can be found in (Liao, 1990)

Substituting equations (3.15)-(3.18) into (3.13), the functional of each element $I_e$ can now be expressed as

$$I_e = \sum_{k=1}^{n} \left( \frac{1}{2} \tilde{g}_e^k K_e^k g_e^k + \tilde{\beta}_e^k \tilde{G}_e \tilde{g}_e^k - \frac{1}{2} \tilde{\beta}_e^k \tilde{H}_e \tilde{\beta}_e^k \right) - \tilde{Q}_e^T \tilde{Q}_e, \quad (3.19a)$$

where

$$K_e^k = \int_{V_{ke}} \tilde{P}^T \tilde{C}^k \tilde{P} dV_{ke},$$

$$\tilde{G}_e^k = \int_{V_{ke}} \tilde{P}^T \tilde{T}^k (\tilde{C}^m \tilde{T}^m \tilde{P} + \tilde{B}) dV_{ke},$$

$$\tilde{H}_e^k = \int_{V_{ke}} \tilde{P}^T \tilde{T}^k \tilde{S}^e \tilde{T}^k \tilde{P} dV_{ke},$$

$$\tilde{Q}_e = \int_{S_{ke}} \tilde{N}^T \tilde{t}^* dS_e. \quad (3.19b)$$

All the volume and surface integral can be carried out by Gauss quadrature. Taking the summation over all the sublamine gives

$$I_e = \frac{1}{2} \tilde{g}_e^T K_e \tilde{g}_e + \tilde{\beta}_e^T G_e \tilde{g}_e - \frac{1}{2} \tilde{\beta}_e^T H_e \tilde{\beta}_e - \tilde{g}_e^T Q_e, \quad (3.20a)$$

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where

\[ q_e = \sum_{k=1}^{n} q_e^k, \]
\[ \beta_e = \sum_{k=1}^{n} \beta_e^k, \]
\[ K_p = \sum_{k=1}^{n} K_p^k, \]
\[ G = \sum_{k=1}^{n} G^k, \]
\[ H = \sum_{k=1}^{n} H^k. \]  

(3.20b)

Operating the variation of \( I_e \) with respect to the nodal displacement \( q_e \) and the stress parameter vector \( \beta_e \), we obtain

\[ \delta I_e = (q_e^T K_p + \beta_e^T G - Q_e^T) \delta q_e + (q_e^T G^T - \beta_e^T H) \delta \beta_e. \]

With the arbitrariness of \( \delta q_e \) and \( \delta \beta_e \), the stationary condition \( \delta I_e = 0 \) provides

\[ q_e^T K_p + \beta_e^T G - Q_e^T = 0, \]  

(3.21a)

and

\[ \beta_e = H^{-1} G q_e. \]  

(3.21b)

Substituting (3.21b) into (3.21a), the element stiffness matrix \( K_e \) for the present multifield finite element model can now be established as

\[ K_e q_e = Q_e, \]  

(3.22a)

where

\[ K_e = K_p + G^T H^{-1} G. \]  

(3.22b)
By executing the element assembly process, the final governing equation for the whole local region of the thick laminate can be written as

\[ K \mathbf{q} = \mathbf{Q}, \quad (3.23a) \]

where

\[ K = \sum_e K_e, \]
\[ \mathbf{Q} = \sum_e \mathbf{Q}_e, \quad (3.23b) \]
\[ \mathbf{q} = \sum_e \mathbf{q}_e, \]

The assembled global equation (3.23) can be solved by any numerical scheme as used in the displacement model to yield the nodal displacements. The whole displacement field of the local region can then be constructed via equation (3.15). The inplane strains are obtained by equation (3.16), while the transverse stresses are provided by equations (3.18) and (3.21b). To find the inplane stresses and transverse strains, equation (3.10) should be used. Thus, the whole local region is analyzed completely.

It should be noted that the present multiphase finite element can be reduced to the partial hybrid stress element (Liao, 1990) by letting \( C^i_\beta = C^\alpha_\beta = 0 \) in equation (3.11b), which leads to \( C^i_{\alpha k} = C^i_k \) and \( C^\alpha_{\alpha k} = 0 \) where \( C^i_k \) is the flexure stiffness matrix named in Liao’s dissertation. However, by permitting \( C^i_\beta \) to be nonzero, the most general anisotropic bodies have been considered in the present element. While in Liao’s element only orthotropic materials are considered. Moreover, the discontinuity of transverse normal stress \( \sigma_z \) is ignored in Liao’s partial hybrid element. Therefore, in practical application the present model is more general and accurate than that proposed by Liao (1990).

3.2 Singlefield Finite Element Models (Displacement-Based)

The singlefield finite element is designed based upon only one kind of functions, displacements, which are functions of \( y \) and \( z \) only. The formulation is the same as that
described in Section 2.1, except that the plane considered now is $yz$ plane not $xy$ plane. To save the space, we will not repeat the formulation and numerical procedure described in Section 2.1. The advantages and disadvantages of this displacement-based element over the multifield element described in the last section have also been discussed in the second paragraph of this chapter.
CHAPTER 4

GLOBAL-LOCAL MODEL

In order to combine the results of global and local regions discussed in Chapters 2 and 3, and show the concept of global-local model, a symmetric laminate with a circular or semi-circular hole under in-plane loading is considered (Figures 2 and 3). An example of [0°/90°], graphite/epoxy considered by 3D finite element method (Lucking, et.al., 1984) is adopted to assess the accuracy of the present method. The material for each lamina used in this analysis is T300/5208 graphite/epoxy of which the material properties are

\[ E_{11} = 145 \text{Gpa}, \quad E_{22} = E_{33} = 10.7 \text{Gpa}, \]

\[ G_{12} = G_{13} = 4.5 \text{Gpa}, \quad G_{23} = 3.6 \text{Gpa}, \]

\[ \nu_{12} = \nu_{13} = 0.31, \quad \nu_{23} = 0.49, \]

where the subscripts 1, 2 and 3 refer to the directions of fiber, transverse and thickness, respectively. Since the multifield finite element can be reduced to the partial hybrid stress finite element proposed by Liao (1990), the generality and accuracy is expected. The main idea in this research is the concept of global-local model. To have a clear understanding of this concept, we choose the special boundary element presented in Section 2.2 for the global region of the hole problems and the displacement-based finite element presented in Section 3.2 for the local region. Similar procedure may be applied to any combination of the numerical scheme presented in this report. As stated previously, use of the displacement-based two-dimensional finite element for the local region is confined to thin composite laminates. Therefore, the examples shown below consider the ratio of thickness to radius to be \( t/R = 1/25 \), which is small enough for the present illustration.

(i) A [0°/90°], laminate with a central hole

In the following analysis, firstly the entire laminate (treated as a global region) is
analyzed in $x_1-x_2$ plane by the special boundary element introduced in this report, then the stresses $\sigma_{ij}$ for each ply are calculated by the stress-strain law of each ply. The negative values of the ply stresses along the hole boundary are used as the prescribed traction for the implementation of the local region. The mesh of the finite element of the local region in $y$-$z$ plane is now extending in $y$-direction from a particular point (of which the interlaminar stresses are examined) on the hole boundary to a distance $\ell$ of the order magnitude $h$ (which is to be determined below), and in $z$-direction from the mid-surface to the top of the laminate since both of the materials and geometry considered here are symmetric with respect to $z = 0$. A typical example of the finite element mesh is shown in Figure 4. The boundary conditions for the finite element meshes are therefore

$$w = 0 \quad \text{for} \quad z = 0,$$

$$\sigma_{33} = \sigma_{32} = \sigma_{31} = 0 \quad \text{for} \quad z = t/2(= 2h),$$

$$\sigma_{22} = -\sigma_{22}^{(0)}, \quad \sigma_{23} = 0, \quad \sigma_{21} = -\sigma_{21}^{(0)} \quad \text{for} \quad y = 0,$$

$$\sigma_{ij} = 0, \quad i, j = 1, 2, 3 \quad \text{for} \quad y = \ell.$$

The determination of the range of boundary layer $\ell$ is dependent of the convergence of stresses. Since the boundary layer stresses decay quickly in $y$-direction from the free edge, i.e., they are very close to zero as $y$ is larger than the laminate thickness, to approximate the condition given in (3.2) only a finite range $\ell$ with the order of laminate thickness need to be considered and the boundary condition along $y = \ell$ is traction-free. To examine the convergence, mesh generations for three different $\ell$'s, $0.5t, 1.5t$ and $4t$, are shown in Figure 5. Note that the highly refined grids used in the region adjacent to the hole edge is due to the highly stress gradients occured in this region. From the results shown in Table 1, it can be seen that even though $\ell$ is taken as $1.5t$ where $t$ represents the thickness of laminate, satisfactory results can be achieved. By comparing the results between mesh 1 and mesh 2, we find that the most discrepancy among three interlaminar stresses is the normal stress
\( \sigma_{zz} \) at mid-surface of the laminate because the decaying speed for this normal stress is slower than those for another two stresses.

It should be noted that including the results in Table 1 the angular distribution of stresses for \( \sigma_{xx}, \sigma_{zz} \) at interface between the first and second plies (0° and 90° plies) and \( \sigma_{zz} \) on the mid-surface are evaluated at \( y = h/20 \). As illustrated in Figures 6, 7 and 8, results of these three stresses are in good agreement with those given in (Lucking, et.al., 1984) where, however, the use of 3D finite element will lead to large computer cost and time-consuming data planing and preparation.

The radial distribution of these three stresses are given in Figures 9-12. They show that the interlaminar stresses \( \sigma_{xx}, \sigma_{zz} \) on the interface plane exhibit a very steep slope near free edge since the nature of singularity exists. The stress gradient for \( \sigma_{zz} \) on the mid-surface near free edge is slightly smaller than the interlaminar stresses at the interface. It is expected that \( \sigma_{zz} \) on the mid-surface converges to a finite value at the hole edge where there is no material discontinuity. Therefore, the value of \( \sigma_{zz} \) at this location is not sensitive to the mesh refinement for the region near the hole edge. Moreover, at a point for \( y = h/20 \), which is very near to the hole edge, almost all the normal stresses for different \( \psi \) at mid-surface are larger than those at interface. This suggests that, in general, a free edge stress singularity occured at the interface between different laminae dominates only a very small region due to its very weak singularity order, which approximate to 0.02 (Ting and Chou, 1981; Wang and Choi, 1982). This singularity oder is small compared to the singularity order of crack, i.e., 0.5.

(ii) A \([0^\circ/90^\circ]\) laminate with a semi-circular hole

The semi-circular hole is located at the edge of a laminate, the plane view of which is shown in Figure 3. If we divide the laminate in the above problem into two parts, each
of them is identical to the laminate in this problem. The boundary condition on $x_2 = 0$ is traction-free. Besides, the properties of the laminate are the same as those in the above problem.

The circumferential distribution of three stresses, i.e., $\sigma_{zz}$, $\sigma_{zz}$ at interface and $\sigma_{zz}$ at midplane are shown in Figure 13. We observe that the shape of distribution for three stresses are almost all the same as the above problem. The main difference between these two problems is in the region near to $x_2 = 0$, where the value of three stresses in this problem is very small compared to those given in the above problem. The largest value of $\sigma_{zz}$ for these two problems are all located at the position $\psi \simeq 80^\circ$, but the value of this problem is larger than those shown in (i).
CHAPTER 5
CONCLUDING REMARKS

A detailed and efficient numerical scheme for the analysis of multi-layer thick laminates has been developed in this research by the application of the global-local concept. The global region is analyzed by the displacement-based finite element for the general two-dimensional problems, or by a special boundary element developed in this research for the hole problems. The local region is then analyzed by the multifield finite element for three-dimensional analysis, or by the singlefield finite element for two-dimensional analysis. The accuracy and convergency of the present global-local concept has also been tested by comparing the results obtained from the 3D finite element. The results show that the present model is really accurate and time saving.
REFERENCES


Zhang, K. and Ueng, C.E.S., A Simplified Approach to Interlaminar Stresses in Com-
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Table 1
mesh 1: Node=303, Element=88

mesh 2: Node=407, Element=120

mesh 3: Node=537, Element=160
The diagram shows the stress $\sigma_{zz}$ normalized by $\sigma_0$ as a function of $\psi$ (degree). There are three curves:

- **Solid line**: $\sigma_{zz}$ (interface)
- **Dashed line**: $\sigma_{zz}$ (midplane)
- **Dashed-dotted line**: $\sigma_{zz}$ (interface)