The Analysis and Design of Composite Laminates with Holes

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ABSTRACT

Because of the anisotropic nature of composite materials, stress concentrations due to the presence of holes or cracks may be substantially higher in composites than for an equivalent metallic structure. Also, fiber composites generally exhibit near-linear elastic behavior to failure. Therefore, the combination of high stress concentration and the absence of ductile yielding means that composites are relatively intolerant of overloads. Because of this, the study on the stress concentrations and strength predictions becomes an important topic on composite laminates. In this study, a systematic approach combining the results of analytical solutions and the technique of boundary element method has been developed in order to analyze the notched laminates efficiently and accurately. The analytical solutions obtained by applying the Stroh's formalism for the problems of two-dimensional anisotropic elastic bodies containing an elliptic hole are totally new in the literature. The fundamental solution for the boundary element introduced has satisfied the traction-free boundary conditions. Hence, the discretization around the elliptic boundary is avoided and relatively coarse meshes can achieve high accuracy. Problems where the hole boundary is not traction free are also solved, such as pin-loaded holes.
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CHAPTER I
INTRODUCTION

Damage tolerant design for composite materials takes into consideration the types of
defects that may, by design, be present in the structure, such as holes, cutouts, and those
defects possibly occurring inadvertently during manufacture or service such as delaminations or cracks.

In the case of holes in composite laminates, several different approaches have been
successfully developed. These methods are the progressive ply failure approach (Yamada
and Sun, 1973), the average stress failure criterion, and the point stress failure criterion
(Whitney and Nuismer, 1974). For the problem of through cracks in a composite laminate,
there have been a variety of approaches yielding accurate results for the reduction in
strength due to the presence of a crack. Each is based on fracture mechanics. The most
widely used methods are: the strain energy density hypothesis (Sih and Chen, 1973), the
compliance approach (Slepetz and Carlson, 1975), the average and point stress criterion
(Nuismer and Whitney, 1975).

To analyze the problems of composites with holes or cracks, the classical lamination
theory is used to model the composites as an anisotropic plate. For two-dimensional
anisotropic elasticity, Stroh’s formalism (Stroh, 1958) which has been shown to be elegant
and powerful, is used to find the analytical solutions for the corresponding infinite bodies.
The stress analyses of two-dimensional finite plates containing a hole or a crack are usually
made by finite element method because of its flexibility to the complicated geometry and
loading. However, very fine mesh or the special singular element should be introduced due
to the singularity behavior near the crack tip or the free edge of interlaminar. In order
to save the computing cost, a special boundary element method in conjunction with the
analytical solution is developed in this research.

The basic assumption of Stroh's formalism is that all components of stresses and displacements in an elastic body depend on \( x_1 \) and \( x_2 \) only, which is the conditions for generalized plane deformation. It can be reduced directly to plane strain problems when out-of-plane displacement is zero. Moreover, it also applicable for the generalized plane stress problems by considering the displacements and stress as the average values through the thickness of the plates (Hwu, 1991). In general, the Stroh's formalism is suitable for the anisotropic material with distinct material eigenvalues. For degenerate materials whose eigenvalues are repeated, such as isotropic materials, the results should be modified in analytical sense (Ting and Hwu, 1988). However, a small perturbation of the material constants can usually make the eigenvalues be distinct and the results can be applied conveniently. In most cases a real form solution which does not contain any eigenvalues and eigenvectors can be obtained and the results are therefore valid for any kind of anisotropic materials even in analytical sense.

The case of anisotropic medium with an elliptic hole under uniform loading or pure bending can be found in Savin (1961) and Lekhnitskii (1968). Using Lekhnitskii's approach, the analytic solutions of infinite media with an elliptic hole subjected to a point force were obtained in series form by Tarn and Chen (1987). In addition, Kamel and Liaw (1989a,b) found the solution for the cases of point forces and concentrated moments by employing the superposition method and Cauchy's integral theorem (Muskhelishvili, 1954). In contrast to the above two approaches, the method developed in this study by applying Stroh's formalism is much simpler and the solution is general with regard to loading and material properties. When the loading is considered to be a point force at arbitrary position, the solution of the associated elasticity problem can be employed as the fundamental solution.
for the boundary element method (Brebbia et al., 1984).

The analytic solutions mentioned above are suitable only for infinite media. Due to the difficulties in satisfying the boundary conditions for finite plate, a boundary element method is adopted in this study to consider the problems for finite plates with a hole. The boundary element method has substantially reduced the computational work since only the boundary need to be discretized. Using boundary element method, the first paper on plane anisotropic problem was presented by Rizzo and Shippy (1970), in which the closed form fundamental solution can be found in (Green, 1941). Since the fundamental solution in boundary integral equation possesses the nature of singularity, poor results are provided at the points near the boundary, particularly on the boundary. This disadvantage may be improved by exactly evaluating the boundary integrals in those element containing singularities, which, nevertheless, appears only in special cases such as isotropic materials or orthotropic materials with constant elements (Maharjeri and Sikarskie, 1986).

For the elastic bodies with holes or cracks, the main concern to us is stresses at the neighborhood of crack tip or on the hole boundary, i.e., stress intensity factor or stress concentration factor. An efficient boundary element method which avoids the discretization of crack or hole surfaces in boundary integral equation can be obtained when the boundary conditions of holes or cracks are satisfied a priori by the fundamental solution. This method proposed by Snyder and Truse (1975) is applied to orthotropic finite plates containing a crack. Subsequently, it was widely used in the analysis of isotropic and anisotropic plates with cracks (Cruse, 1978; Murakami, 1978; Clements and Haselgrove, 1983; Ang and Clements, 1986, 1987; Ang, 1987, Tan and Bigelow, 1988; Mews and Kuhn, 1988). M.M.S. and Altiero (1979a, 1979b) employed this concept to consider the elastic isotropic body with an arbitrary shape contour and also treated the case of crack which is modelled as
a very narrow ellipse. Later, this boundary element method was further applied to the isotropic problem with time-dependent inelastic deformation (Mukherjee and Morjaria, 1981a, b). The cases which are valid for monoclinic materials and inplane loading were considered by Tarn and Chen (1987).

In this study, we use boundary element method, in which the Green's function satisfies the traction-free condition on an elliptic hole in an infinite anisotropic medium, to consider finite plates with a hole or crack. We also examine the problems where the elliptic holes are loaded by pins. By careful superposition of stresses, the numerical procedure for the traction-free hole problem can be used to solve the pin-loaded hole problem. Numerical results for various specific problems are obtained.
CHAPTER II
STRESS ANALYSES OF COMPOSITE LAMINATES

2.1 Classical Lamination Theory

Consider now a laminate comprising \( n \) plies and denote the angle between the fiber direction in the \( k^{th} \) ply and \( x_1 \) structural axis by \( \theta_k \) which is arbitrary. It is assumed that, when the plies are molded into the laminate, a rigid bond (of infinitesimal thickness) is formed between adjacent plies. As a consequence of this assumption, it follows that under plane stress conditions the strains are the same at all points on a line through the thickness (i.e. they are independent of \( z \)). Therefore, the composite laminate can be treated as a homogeneous membrane having stiffness properties of an anisotropic material. It should be noted that the stresses under this assumption are only the average values over the laminate thickness.

Under the assumption stated previously, the elastic matrix \( C_{ij} \), which is the contracted notation for the elasticity tensor \( C_{ijkl} \), used for the analysis of composite laminates can be calculated by

\[
C_{ij} = \frac{1}{h} \sum_{k=1}^{n} C_{ij}^{(k)} h_k \quad , \quad i,j = 1,2,\cdots,6,
\]

where \( h_k \) is the thickness of \( k^{th} \) lamina, \( h \) is the thickness of the laminate and \( C_{ij}^{(k)} \) is the elastic matrix of \( k^{th} \) lamina.

Actually, the transverse shear stresses are developed under in-plane loadings. It follows that there is some boundary layer around the interlaminar free edge. This boundary layer would be expected to extend in from the edges a distance of the order of the laminate thickness. In the boundary layer, the above assumption is not applicable and a three-dimensional analysis is required. The matter is of more than academic interest since faults, such as delaminations, are prone to originate at the free edges of laminates because
of the above effect. However, as far as the stress concentration around the through holes are of more concern the simple laminate theory stated above can still be applied for thin plate and the results are reliable as compared to the three-dimensional analysis. In the following, attention is then focused on the analysis of two-dimensional anisotropic plates.

2.2 Two-Dimensional Anisotropic Elasticity

For two-dimensional anisotropic elasticity, there are two different formulation in the literature. One is the Lekhnitskii’s approach (1968) which starts with the equilibrated stress functions then compatibility equations, the other is Stroh’s formalism (1958) which starts with the compatible displacements then equilibrium equations. The equivalency of these two formulations has been discussed in (Suo, 1990). In this paper, we follow Stroh’s formalism due to its elegance and simplicity and the notation in (Hwu and Ting, 1989) is employed. In a fixed rectangular coordinate system \( x_i, i = 1, 2, 3 \), let \( u_i, \sigma_{ij}, \epsilon_{ij} \) be, respectively, the displacement, stress and strain. With body forces neglected, the strain-displacement equations, the stress-strain laws, and the equations of equilibrium are

\[
\begin{align*}
\epsilon_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i}) , \\
\sigma_{ij} &= C_{ijkl} \epsilon_{kl} , \\
\sigma_{ij,j} &= C_{ijkl} u_{k,aj} = 0 ,
\end{align*}
\]

where repeated indices imply summation, a comma stands for differentiation and \( C_{ijkl} \) are the elastic constants which are assumed to be fully symmetric and positive definite. If we consider a two-dimensional deformation in which \( u_i, i = 1, 2, 3 \), depend on \( x_1 \) and \( x_2 \) only, the displacement \( u_i \) can be written as

\[
u_i = a_i f(z), \quad z = x_1 + px_2. \tag{2.4}
\]
By substituting (2.4) into (2.3), the general solution for $u_i$ can be written in matrix notation as

$$u = \sum_{\alpha=1}^{6} g_{\alpha}f_{\alpha}(z_{\alpha}), \quad z_{\alpha} = x_1 + p_{\alpha}x_2, \quad (\alpha \text{ not summed}), \quad (2.5)$$

in which $f_1, f_2, \cdots$ are arbitrary functions of their arguments and $p_{\alpha}$ and $z_{\alpha}$ are the eigenvalues and eigenvectors of the following eigenrelation:

$$\{Q + p(R + R^T) + p^2T\}z = 0. \quad (2.6)$$

In (2.6) the superscript $T$ stands for the transpose and $Q, R, T$ are the $3 \times 3$ real matrices given by

$$Q_{ik} = C_{i1k1}, \quad R_{ik} = C_{i1k2}, \quad T_{ik} = C_{i2k2}. \quad (2.7)$$

We see that $Q$ and $T$ are symmetric and positive definite if the strain energy is positive (Esbelby, Read and Shochley, 1953). Because the eigenvalue $p$ of (2.6) cannot be real if the strain energy is positive, we can obtain three pairs of complex conjugates for $p$. Let

$$p_{\alpha+3} = \bar{p}_\alpha, \quad \text{Im}(p_{\alpha}) > 0, \quad \alpha = 1, 2, 3,$$

where an overbar denotes the complex conjugate and Im stands for the imaginary part.

We then have

$$z_{\alpha+3} = \bar{z}_\alpha, \quad \alpha = 1, 2, 3.$$

For the displacement $\tilde{u}$ to be real, we let

$$f_{\alpha+3} = \bar{f}_\alpha, \quad \alpha = 1, 2, 3,$$

and (2.5) becomes

$$\tilde{u} = 2\text{Re}\left\{\sum_{\alpha=1}^{3} g_{\alpha}f_{\alpha}(z_{\alpha})\right\}, \quad (2.8)$$
in which $Re$ stands for the real part.

Introducing the vector

$$\tilde{b} = (R^T + pT)\tilde{a} = -\frac{1}{p}(Q + pR)\tilde{a},$$  \hspace{1cm} (2.9)

where the second equality comes from (2.6), the stresses $\sigma_{ij}$ obtained by substituting (2.5) into (2.1) and (2.2) can be written as

$$\sigma_{i1} = -\phi_{i,2}, \quad \sigma_{i2} = \phi_{i,1},$$  \hspace{1cm} (2.10)

where $\phi$ is the stress function

$$\phi = 2Re \left\{ \sum_{\alpha=1}^{3} b_{\alpha} f_{\alpha}(z_{\alpha}) \right\},$$  \hspace{1cm} (2.11)

and $\tilde{b}_{\alpha}$ is related to $\tilde{a}_{\alpha}$ through (2.9). More generally, if $\tilde{t}$ is the surface traction at a point on a curve boundary, then

$$\tilde{t} = \partial \phi / \partial s,$$  \hspace{1cm} (2.12)

where $s$ is the arc length measured along the curved boundary in the direction such that, when one faces the direction of increasing $s$, the material is located on the right-hand side. We see that (2.10) are special cases of (2.12) when the boundary is a plane parallel to the $x_2$-axis or the $x_1$-axis.

With similar reason as Suo (1990) that whether a function is analytic is not affected by different arguments $z_{\alpha} = x_1 + p_\alpha x_2$, $\alpha = 1, 2, 3$, for (2.8) and (2.11) another solution form appropriate for the method of analytic continuation is written as

$$\tilde{u} = A\tilde{f}(z) + \overline{A\tilde{f}(z)},$$  \hspace{1cm} (2.13a)

$$\phi = B\tilde{f}(z) + \overline{B\tilde{f}(z)},$$
where
\[ A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \]
\[ B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}, \]
\[ \mathbf{f}(z) = \begin{bmatrix} f_1(z) & f_2(z) & f_3(z) \end{bmatrix}^T. \]
\hspace{2cm} \tag{2.13b} \]
Note that the argument of each component function of \( \mathbf{f}(z) \) is written as \( z = x_1 + px_2 \) without referring to the associated eigenvalues \( p_\alpha \). Once the solution of \( \mathbf{f}(z) \) is obtained for a given boundary value problem, a replacement of \( z_1, z_2 \) or \( z_3 \) should be made for each component function to calculate field quantities from (2.8) and (2.11).

In many applications, \( f_1, f_2, f_3 \) have the same function form
\[ f_\alpha(z_\alpha) = q_\alpha f(z_\alpha), \quad \alpha \text{ not summed}, \]
where \( q_\alpha, \alpha = 1, 2, 3, \) are arbitrary complex constants. Equations (2.8) and (2.11) can then be written as
\[ \mathbf{u} = 2\text{Re}\{ A \ll f(z_\alpha) \gg q_\alpha \}, \]
\[ \phi = 2\text{Re}\{ B \ll f(z_\alpha) \gg q_\alpha \}, \]
\hspace{2cm} \tag{2.14} \]
in which \( q_\alpha \) is the \( 3 \times 1 \) matrix whose elements are \( q_\alpha, \alpha = 1, 2, 3 \) and the angular bracket stands for the diagonal matrix, i.e., \( \ll f_\alpha \gg = \text{diag}\{ f_1 \ f_2 \ f_3 \} \), which will be used throughout this report.

### 2.3 The Sextic Formalism of Stroh

The two equations in (2.9) can be written in a standard eigenrelation
\[ N\xi = p\xi, \]
\[ N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_T \end{bmatrix}, \quad \xi = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \]
\[ N_1 = -N^{-1}R_T, \]
\[ N_2 = T^{-1}, \]
\[ N_3 = R_T^{-1}R_T - Q, \]
\hspace{2cm} \tag{2.15, 2.16, 2.17} \]
where $N_2$ and $N_3$ are symmetry and $N_2$ is positive definite. With the definition in (2.13b), the matrices $\tilde{A}$ and $\tilde{B}$ satisfy the following orthogonality relations as (Stroh, 1958; Ting, 1986)

\begin{align}
\tilde{A}^T \tilde{B} + \tilde{B}^T \tilde{A} &= I = \tilde{A}^T \tilde{B} + \tilde{B}^T \tilde{A}, \\
\tilde{A}^T \tilde{B} + \tilde{B}^T \tilde{A} &= 0 = \tilde{B}^T \tilde{A} + \tilde{A}^T \tilde{B}, \\
\tilde{A}\tilde{A}^T + \tilde{B}\tilde{B}^T &= 0 = \tilde{B}\tilde{B}^T + \tilde{A}\tilde{A}^T, \\
\tilde{B}\tilde{A}^T + \tilde{A}\tilde{B}^T &= I = \tilde{A}\tilde{B}^T + \tilde{B}\tilde{A}^T, 
\end{align}

where $I$ is the unit matrix. It follows from (2.18c,d) that three real matrices $\tilde{H}, \tilde{S},$ and $\tilde{L}$ can be defined as (Ting, 1986)

\begin{align}
\tilde{H} &= 2i \tilde{A}\tilde{A}^T, \\
\tilde{L} &= -2i \tilde{B}\tilde{B}^T, \\
\tilde{S} &= -i(2\tilde{A}\tilde{B}^T - I). 
\end{align}

It can be shown that $\tilde{H}$ and $\tilde{L}$ are symmetric and positive definite (Chadwick and Smith, 1977).

2.4 Identities

To generalize the eigenrelation (2.15) one consider another formalism by assuming that

\begin{align}
\psi = a f(z \cdot \zeta + p(\theta)z \cdot m),
\end{align}

where the symbol "$\cdot$" denote the inner product of two vectors and

\begin{align}
\zeta^T(\theta) &= (\cos \theta, \sin \theta, 0) \\
m^T(\theta) &= (\sin \theta, \cos \theta, 0)
\end{align}
and \( \theta \) is an arbitrary real parameter. With (2.20), equation (2.7) is replaced by

\[
Q_{ik} = C_{ijks} n_j n_k,
\]

\[
R_{ik} = C_{ijks} n_j m_k,
\]

\[
T_{ik} = C_{ijks} m_j m_k,
\]

and (2.15)-(2.17) become

\[
\overline{N}(\theta) \hat{\xi} = p(\theta) \hat{\xi},
\]

\[
\overline{N}(\theta) = \begin{bmatrix} \overline{N}_1(\theta) & \overline{N}_2(\theta) \\ \overline{N}_3(\theta) & \overline{N}_T(\theta) \end{bmatrix}, \quad \hat{\xi} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}
\]

\[
\overline{N}_1(\theta) = - T^{-1}(\theta) \overline{R}^T(\theta),
\]

\[
\overline{N}_2(\theta) = T^{-1}(\theta),
\]

\[
\overline{N}_3(\theta) = \overline{R}(\theta) T^{-1}(\theta) \overline{R}^T - \overline{Q}(\theta).
\]

As before, there are six eigenvalues \( p_\alpha(\theta) \), \( \alpha = 1, 2, \ldots, 6 \) which come in three pairs of complex conjugate. It can be shown (Chadwick and Smith, 1977) that when \( \overline{N}(\theta) \) is simple or semi-simple \( \hat{\xi} \) is independent of \( \theta \). It should be noted that (2.23) to (2.25) can reduce to (2.15) to (2.17) when \( \theta = 0 \). The eigenvalue \( p_\alpha(\theta) \) are related to \( p_\alpha \) by (Ting, 1982)

\[
p_\alpha(\theta) = \frac{p_\alpha \cos \theta - \sin \theta}{p_\alpha \sin \theta + \cos \theta}.
\]

Equation (2.23) written for \( p(\theta) = p_1(\theta), p_2(\theta), p_3(\theta) \) can be combined into one compact form as

\[
\overline{N}(\theta) \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \ll p_\alpha(\theta) \gg \\ B \ll p_\alpha(\theta) \gg \end{bmatrix},
\]

where \( \overset{\sim}{A} \) and \( \overset{\sim}{B} \) are defined in (2.13b). Postmultiplying both sides of (2.27) by \( \overset{\sim}{B}^T, \overset{\sim}{A}^T \) and using (2.19), we can obtain

\[
2 \begin{bmatrix} \overset{\sim}{A} \ll p_\alpha(\theta) \gg \overset{\sim}{B}^T \\ \overset{\sim}{B} \ll p_\alpha(\theta) \gg \overset{\sim}{A}^T \end{bmatrix} = \overline{N}(\theta) \begin{bmatrix} \overset{\sim}{I} - i\overset{\sim}{S} \\ i\overset{\sim}{L} \overset{\sim}{I} - i\overset{\sim}{S}^T \end{bmatrix}.
\]
Substituting (2.24) into (2.28), we have the following identities:

\[ 2 \tilde{A} \ll p_\alpha(\theta) \gg \tilde{B}^T = \tilde{N}_1(\theta) - i[\tilde{N}_1(\theta) \tilde{S} - \tilde{N}_2(\theta) \tilde{L}] \]
\[ = \tilde{N}_1(\theta) - i[\tilde{S}\tilde{N}_1(\theta) + \tilde{H}\tilde{N}_2(\theta)], \]

\[ 2 \tilde{A} \ll p_\alpha(\theta) \gg \tilde{A}^T = \tilde{N}_2(\theta) + i[\tilde{N}_1(\theta) \tilde{H} - \tilde{N}_2(\theta) \tilde{S}^T], \]

\[ 2 \tilde{B} \ll p_\alpha(\theta) \gg \tilde{L}^T = \tilde{N}_3(\theta) - i[\tilde{S}\tilde{N}_3(\theta) - \tilde{N}_1^T(\theta) \tilde{L}] . \] (2.29)

If we postmultiply both sides of (2.27) by \( \tilde{L}^{-1}, \tilde{A}^{-1} \) and using (Ting, 1983)

\[ \tilde{A}\tilde{B}^{-1} = -(\tilde{S} + i \tilde{L})\tilde{L}^{-1} = \tilde{L}^{-1}(\tilde{S}^T - i \tilde{L}), \]
\[ \tilde{B}\tilde{A}^{-1} = -(\tilde{S}^T + i \tilde{L})\tilde{H}^{-1} = -\tilde{H}^{-1}(\tilde{S} - i \tilde{L}). \] (2.30)

we can obtain the following identities

\[ \tilde{A} \ll p_\alpha(\theta) \gg \tilde{A}^{-1} = (\tilde{N}_1(\theta) - \tilde{N}_2(\theta)\tilde{H}^{-1}\tilde{S}) + i\tilde{N}_2(\theta)\tilde{H}^{-1}, \]
\[ \tilde{A} \ll p_\alpha(\theta) \gg \tilde{B}^{-1} = (\tilde{N}_3(\theta) + \tilde{N}_1(\theta)\tilde{L}^{-1}\tilde{S}^T) - i\tilde{N}_1(\theta)\tilde{L}^{-1}, \]
\[ \tilde{B} \ll p_\alpha(\theta) \gg \tilde{B}^{-1} = (\tilde{N}_3^T(\theta) - \tilde{N}_2(\theta)\tilde{S}\tilde{L}^{-1}) - i\tilde{N}_2(\theta)\tilde{L}^{-1}, \]
\[ \tilde{B} \ll p_\alpha(\theta) \gg \tilde{A}^{-1} = (\tilde{N}_3(\theta) - \tilde{N}_1^T(\theta)\tilde{H}^{-1}\tilde{S}) - i\tilde{N}_1^T(\theta)\tilde{H}^{-1}. \] (2.31)

The identities given in (2.30) and (2.31) will be useful in obtaining a real form solution to two-dimensional anisotropic elasticity problems.

Barnett and Lothe (1973) proposed an alternative expression for \( \tilde{S}, \tilde{H}, \tilde{L} \) defined in (2.19) as

\[ \tilde{S} = \frac{1}{\pi} \int_0^\pi \tilde{N}_1(\theta) d\theta, \]
\[ \tilde{H} = \frac{1}{\pi} \int_0^\pi \tilde{N}_2(\theta) d\theta, \]
\[ \tilde{L} = -\frac{1}{\pi} \int_0^\pi \tilde{N}_3(\theta) d\theta. \] (2.32)

In this way, the need to determine the eigenvalues \( p_\alpha \) and eigenvectors \( \tilde{A} \) and \( \tilde{B} \) is circumvented and hence (2.32) is valid regardless of \( \tilde{N} \) is simple, semisimple or non-semisimple.
CHAPTER III
INFINITE BODIES WITH HOLES

In this chapter, an infinite anisotropic body with an elliptic hole under arbitrary loading is considered. The applied loading is located either in the medium or on the hole boundary. With the general solutions obtained, several examples are solved explicitly for the purpose of illustration. Besides, some of the results such as the case of uniform loading can be used as a check for the correctness of the present results. Moreover, the results of the point loading conditions can be used as the fundamental solutions of the boundary element method described in the next chapter for the finite bodies with a hole or crack.

3.1 Infinite Bodies with Traction-Free Holes

Consider an elliptic hole embedded in an infinite anisotropic matrix (Figure 1). The contour of the elliptic boundary is represented by

\[ x_1 = a \cos \psi, \quad x_2 = b \sin \psi, \]  \hspace{1cm} (3.1)

where \( \psi \) is a real parameter and \( a, \ b \) are lengths of semi-axes of ellipse. It is known that the transformation function,

\[ z_\alpha = \frac{1}{2} \{ (a - ibp_\alpha)\zeta_\alpha + (a + ibp_\alpha)\frac{1}{\zeta_\alpha} \}, \]  \hspace{1cm} (3.2a)

or

\[ \zeta_\alpha = \frac{z_\alpha + \sqrt{z_\alpha^2 - a^2 - p_\alpha^2 b^2}}{a - ip_\alpha b}, \]  \hspace{1cm} (3.2b)

will map the points \( z_\alpha \) along the elliptic boundary onto a unit circle in \( \zeta_\alpha \)-domain.

For the hole problems, the general solutions shown in (2.13) can be written in terms of the variables \( \zeta_\alpha \) as

\[ \psi = \mathcal{A}[\mathcal{L}_\alpha(\zeta) + \mathcal{F}(\zeta)] + \overline{\mathcal{A}}[\overline{\mathcal{L}_\alpha(\zeta)} + \overline{\mathcal{F}(\zeta)}] \]  \hspace{1cm} (3.3)

\[ \phi = \mathcal{B}[\mathcal{L}_\alpha(\zeta) + \mathcal{F}(\zeta)] + \overline{\mathcal{B}}[\overline{\mathcal{L}_\alpha(\zeta)} + \overline{\mathcal{F}(\zeta)}], \]
where \( \mathbf{f}_o \) represents the function associated with the unperturbed elastic field which is related to the solutions of homogeneous media and is holomorphic in the entire domain except some singular points such as the points under concentrated forces or dislocations, and the points at zero or infinity. \( \mathbf{f} \) is the function corresponding to the perturbative field of matrix and is holomorphic in region \( S \) except some singular points. \( S \) denotes the region occupied by the matrix. Hence, in \( \zeta_o \)-plane, \( S \) is the region outside the unit circle.

For a given loading condition, \( \mathbf{f}_o \) can be obtained immediately since it is related to the solutions of homogeneous media. However, it is not necessary to be exactly the same as the solutions of homogeneous media. The choices of \( \mathbf{f}_o \) depend on the convenience in calculation. The final solutions for the stresses and deformations in the entire domain will not be influenced by the choices of \( \mathbf{f}_o \). To have a better understanding about the choices, two special examples are discussed in the following.

(a) A dislocation \( \hat{\mathbf{b}} \) or point force \( \hat{\mathbf{p}} \) at \( z_o = \hat{z}_o \)

Consider a dislocation line in the direction perpendicular to \( x_1x_2 \) plane with Burger vector \( \hat{\mathbf{b}} \), and a point force uniformly distributed along \( x_3 \)-axis with force per unit length \( \hat{\mathbf{p}} \). Both singularities are at the point \( (\hat{x}_1, \hat{x}_2) \). If \( \mathbf{f}_o \) is chosen to represent exactly the solutions of homogeneous media, it may be written as

\[
\mathbf{f}_o(\zeta) = \ll \log(z_o - \hat{z}_o) \gg \mathbf{q} \quad ,
\]

where \( \mathbf{q} \) is to be determined in terms of \( \hat{\mathbf{b}} \) and \( \hat{\mathbf{p}} \). Using the condition that around any circle enclosing the point \( (\hat{x}_1, \hat{x}_2) \), the total displacements and forces resulting from \( \mathbf{f}_o(\zeta) \) are \( \hat{\mathbf{b}} \) and \( \hat{\mathbf{p}} \). Hence,

\[
2\text{Re}\{i\mathbf{p} \mathbf{q}\} = \hat{\mathbf{p}}/2\pi \quad ,
\]

\[
2\text{Re}\{i\mathbf{A} \mathbf{q}\} = \hat{\mathbf{b}}/2\pi \quad .
\]
To find $q$ from (3.5), we use the orthogonality relations given in (2.18a,b) and obtain

$$q = \frac{1}{2\pi i} (A^T \hat{p} + B^T \hat{b}).$$

(3.6)

However, it is inconvenient in calculation when our general solution is expressed in terms of the variable $\zeta$ not $z$. An alternative choice for $f_0$ is

$$f_0(\zeta) = \ll \log(\zeta - \hat{\zeta}) \gg q,$$

(3.7)

where $q$ is the same as (3.6). This expression is more convenient than the one given in (3.4). Moreover, it also reflects the singularity characteristics of the original problems.

(b) Uniform loading applied at infinity

The exact solution corresponding to the homogeneous media is (Ting, 1988)

$$f_0(\zeta) = \ll z \gg q$$

$$= \frac{1}{2} \ll a - ib\alpha \gg \ll \zeta \gg \frac{a + ib\alpha}{a - ib\alpha} \gg \ll \zeta^{-1} \gg q$$

(3.8a)

where

$$q = A_2^T t_2^\infty + B_2^T \mathcal{E}_2^\infty,$$

$$t_2^\infty = \begin{pmatrix} \sigma_{12}^\infty \\ \sigma_{22}^\infty \\ \sigma_{32}^\infty \end{pmatrix}, \quad \mathcal{E}_2^\infty = \begin{pmatrix} \varepsilon_{11}^\infty \\ \varepsilon_{12}^\infty \\ 2\varepsilon_{13}^\infty \end{pmatrix}.$$  

(3.8b)

$\sigma_{ij}^\infty, \varepsilon_{ij}^\infty$ are the constant stresses and strains induced by the uniform loading applied at infinity. An alternative choice may be provided by

$$f_0(\zeta) = \ll \zeta \gg q_0,$$

(3.8c)

where

$$q_0 = \frac{1}{2} \ll a - ib\alpha \gg q,$$

(3.8d)

and $q$ is the same as (3.8b). The infinity loading conditions are satisfied for both of the choices. The one given in (3.8c) is not a solution for uniform stress distribution, which
can be seen from the transformation function (3.2). However, in calculation (3.8c) is
more convenient than (3.8a), because the singular points of (3.8c) is at infinity only while
singularities occur at zero and infinity for (3.8a). In summary, any type of $f_o$ which can
represent the singular behavior including the point at infinity are all the proper choices.

Based upon the above discussion, we know that if all the singular points of the physical
domain $z_\alpha$ are considered to be located in the matrix, for different choices the complex
function $f_o$ associated with the general loading conditions may be expressed as follows.

(i) By Taylor's expansion,

$$f_o(\zeta) = \sum_{k=0}^{\infty} \varepsilon_k \zeta^k,$$

where

$$\varepsilon_k = \frac{f_o^{(k)}(0)}{k!} = \frac{1}{2\pi i} \int_C \frac{f_o(\zeta)}{\zeta^{k+1}} d\zeta.$$

$\zeta$ belongs to a bounded region where $f_o$ is holomorphic. The cases of $f_o = \ll \log(z_\alpha - \zeta_\alpha) \gg g$ and $f_o = \ll \zeta_\alpha \gg g_o$ belong to this category.

(ii) By Laurent's expansion,

$$f_o(\zeta) = \sum_{k=-\infty}^{\infty} \varepsilon_k \zeta^k,$$

where

$$\varepsilon_{-k} = \frac{\varepsilon_k}{\zeta}, \quad \varepsilon_{k} = \frac{1}{2\pi i} \int_C \frac{f_o(\zeta)}{\zeta^{k+1}} d\zeta.$$

$\zeta$ belongs to an annual ring where $f_o$ is holomorphic. The cases of $f_o = \ll \log(z_\alpha - z_\alpha) \gg g$ and $f_o = \ll z_\alpha \gg g_o$ belong to this category.

When the hole is traction-free, $\bar{q} = 0$ along the hole boundary, which leads to

$$\mathcal{B} \mathcal{f}(\sigma) + \overline{\mathcal{B} f_o(\sigma)} = -\overline{\mathcal{B} f(\sigma)} - \mathcal{B} f_o(\sigma)$$

(3.11)
if \( f_\alpha \) belongs to case (i). By the method of analytic continuation (Muskhelishvili, 1953) we find that

\[
\mathcal{f} (\zeta) = -\mathcal{B}^{-1} \mathcal{B}_\mathcal{f}_\alpha (\frac{1}{\zeta}).
\]

(3.12)

If one is interested in the hoop stress along the hole boundary, calculation may be performed by using the field solution of the matrix. The hoop stress based upon the coordinate system \((\hat{n}, \hat{m})\) which are, respectively, the unit vectors tangent and normal to the interface boundary, is obtained as (Hwu and Ting, 1989)

\[
\sigma_{nn} = -\hat{n}^T (\theta) \hat{\phi},
\]

(3.13a)

where

\[
\hat{n}^T (\theta) = (\cos \theta, \sin \theta, 0), \quad \hat{m}^T (\theta) = (- \sin \theta, \cos \theta, 0),
\]

(3.13b)

and the angle \( \theta \) is directed counterclockwisely from the positive \( x_1 \)-axis to the direction of \( \hat{n} \). The relation between \( \theta \) and \( \psi \) is

\[
\rho \cos \theta = a \sin \psi, \quad \rho \sin \theta = -b \cos \psi,
\]

(3.14)

\[
\rho = \sqrt{a^2 \sin^2 \psi + b^2 \cos^2 \psi}.
\]

The evaluation of \( \hat{\phi}_m \) can be performed by using chain rule as

\[
\frac{\partial f}{\partial m} = \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial \psi} \frac{\partial \psi}{\partial \zeta} \left[ \frac{\partial z_\alpha}{\partial x_1} \frac{\partial x_1}{\partial m} + \frac{\partial z_\alpha}{\partial x_2} \frac{\partial x_2}{\partial m} \right],
\]

(3.15a)

where

\[
\zeta_\alpha = e^{i\psi}, \quad \frac{\partial \zeta_\alpha}{\partial \psi} = ie^{i\psi}, \quad \frac{\partial z_\alpha}{\partial \psi} = -\rho (\cos \theta + p_\alpha \sin \theta),
\]

\[
\frac{\partial x_1}{\partial m} = -\sin \theta, \quad \frac{\partial x_2}{\partial m} = \cos \theta,
\]

(3.15b)

\[
\frac{\partial z_\alpha}{\partial x_1} = 1, \quad \frac{\partial z_\alpha}{\partial x_2} = p_\alpha,
\]
along the hole boundary. By using the results of (3.12) and applying (3.15) we have

\[ \phi_m = -\frac{4}{\rho} N_3(\theta) L^{-1} \sum_{k=1}^{\infty} Re\{ke^{ik\gamma} B e_k\}, \tag{3.16a} \]

or

\[ \phi_m = -\frac{4}{\rho} N_3(\theta) L^{-1} \sum_{k=1}^{\infty} Re\{ke^{ik\gamma} B e_k\}, \tag{3.16b} \]

when \( f_o \) is expressed by the Taylor's expansion as (3.9). During the derivation of equation (3.16), one should be very careful about the \( f_o(\zeta) \) given in (3.12), whose argument of each component function should be replaced by \( \zeta_1, \zeta_2 \) and \( \zeta_3 \), respectively. Moreover, the third identity in (2.31) has been used. If \( f_o \) belongs to case (ii), similar approach can be applied and the results are

\[ f_o(\zeta) = -B^{-1} \sum_{k=1}^{\infty} \{B e_k + \sum_{k} e_k\} \zeta^{-k}. \tag{3.17} \]

The expression for the hoop stress is the same as (3.13) and (3.16b).

### 3.1.1 Point forces

When the medium is subjected to a point force \( \tilde{p} \) located at \( \zeta = (x, \bar{y}) \), the complex function \( f_o(\zeta) \) can be written explicitly by substituting (3.7) into (3.12) with the understanding that the subscripts of \( \zeta \) are dropped before the multiplication of matrices and a replacement of \( \zeta_\alpha \) should be made for each component function of \( f_o(\zeta) \) after the multiplication of matrices. The result is

\[ f_o(\zeta) = \sum_{k=1}^{3} \log(\zeta^{-1} - \zeta_k) \Rightarrow B^{-1} \sum_{k} I_k A^T \tilde{p} / 2\pi i \tag{3.18} \]

where

\[ I_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]
The derivative \( \phi_{,m} \) shown in (3.16) used to calculate the hoop stress can be reduced to
\[
\phi_{,m} = \frac{2}{\pi \rho} N_3(\theta) L^{-1} \Re \left\{ \hat{B} \ll i e^{i\psi} (e^{i\psi} - \zeta)\right\} \hat{\eta}.
\] (3.19)

### 3.1.2 Uniform loads at infinity

If the complex function \( f_o(\zeta) \) associated with the unperturbed elastic field is chosen as those shown in (3.8a), it belongs to the case (ii). The function \( f(\zeta) \) corresponding to the perturbative field of matrix is then obtained from (3.17) with
\[
\varepsilon_1 = \frac{1}{2} \ll a - ib\alpha \gg \gg, \quad \varepsilon_k = 0, \quad k = 2, 3, \ldots \infty.
\] (3.20)

The final simplified result is
\[
f(\zeta) = -\frac{1}{2} \ll \zeta^{-1} \gg B^{-1}(a t_2^\infty - ib t_1^\infty),
\] (3.21)
which can be proved to be identical to those given in Hwu and Ting (1989). In the derivation of (3.21), the first and third identities given in (2.29) with \( \theta = 0 \), have been used. Moreover, the identity (Ting, 1988)
\[
N_1 t_2^\infty + N_3 \varepsilon_1^\infty = - t_1^\infty,
\] (3.22)
and (2.19) are also needed.

As stated previously, \( f_o \) can also be chosen as
\[
f_o(\zeta) = \ll \zeta \gg \gg, \quad \gg = \frac{1}{2} \ll a - ib\alpha \gg \gg.
\]

For this choice, function \( f(\zeta) \) should be found by using (3.12) instead of (3.17) since \( f_o(\zeta) \) now belongs to the case (i). By careful derivation, one can prove that the final results of
\( f_o + f_1 \) are the same for different choices of \( f_o \). A real form solution for the hoop stress along the hole boundary can be obtained by substituting (3.20) into (3.16b) and applying the identities given in (2.19), (2.29), (3.22), which can also be proved to be identical to those shown in Hwu and Ting (1989).

3.2 Infinite Bodies with Arbitrary Loading Applied on the Hole

If the loading is applied on the hole boundary, the boundary condition \( \dot{\varphi} = 0 \) in Section 3.1 along the hole boundary is no longer valid. It is expedient to express the loading along the hole boundary by using Fourier expansion. As a result, the solution can be obtained in a series form. It is worth being mentioned that real form solution of the stresses along the hole boundary will be found.

Any given arbitrary loadings \( \dot{\tilde{t}}_m \) applied on the elliptical hole boundary, as shown in Figure 1, can be expressed by Fourier expansion as

\[
\rho \dot{\tilde{t}}_m = \zeta_0 + \sum_{k=1}^{\infty} \left( c_k \cos k\psi + d_k \sin k\psi \right), \tag{3.23a}
\]

where

\[
\begin{align*}
\zeta_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \rho \dot{\tilde{t}}_m d\psi, \\
\zeta_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} \rho \dot{\tilde{t}}_m \cos k\psi d\psi, \\
d_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} \rho \dot{\tilde{t}}_m \sin k\psi d\psi.
\end{align*} \tag{3.23b}
\]

To satisfy the above loading condition, the stress function \( \dot{\varphi} \) can be assumed as

\[
\dot{\varphi} = 2\text{Re} \left\{ \mathcal{B} \ll \log \zeta_{\alpha} \gg q_0 \right\} + 2 \sum_{k=1}^{\infty} \text{Re} \left\{ \mathcal{B} \ll \zeta_{-k}^{-1} \gg q_k \right\}, \tag{3.24}
\]

in which the solution form of (2.14) is applied. To get real form expression, we replace the complex constant \( q_k \) by

\[
q_k = A^T q_k + B^T \tilde{h}_k, \quad k = 0, 1, 2, \ldots \infty, \tag{3.25}
\]
where $g_k$ and $h_k$ are real. Using (3.15) and (2.19), the traction $\tilde{t}_m = \tilde{\varphi}_a$ along the hole boundary is now obtained as

$$\rho \tilde{t}_m = - (\tilde{S}^T g_0 - \tilde{L} \tilde{h}_0) + \sum_{k=1}^{\infty} k \left\{ (\tilde{S}^T g_k - \tilde{L} \tilde{h}_k) \cos k\psi + g_k \sin k\psi \right\} .$$

(3.26)

By applying the boundary condition $\tilde{t}_m = \tilde{t}_m$, comparison between (3.23a) and (3.26) leads to

$$\tilde{S}^T \tilde{g}_0 - \tilde{L} \tilde{h}_0 = - \tilde{c}_0 ,$$

$$\tilde{S}^T \tilde{g}_k - \tilde{L} \tilde{h}_k = \frac{1}{k} \tilde{c}_k ,$$

(3.27)

$$\tilde{g}_k = \frac{1}{k} \tilde{d}_k , \quad k = 1, 2, \ldots, \infty .$$

The requirement of single-valued displacement gives

$$2 \text{Re} \{ i \tilde{A} \tilde{g}_0 \} = \tilde{H} \tilde{g}_0 + \tilde{S} \tilde{h}_0 = 0 .$$

(3.28)

Combining (3.27) and (3.28), we obtain

$$\tilde{g}_0 = \tilde{S}^T \tilde{c}_0 , \quad \tilde{h}_0 = \tilde{H} \tilde{c}_0 ,$$

$$\tilde{g}_k = \frac{1}{k} \tilde{d}_k , \quad \tilde{h}_k = \frac{1}{k} L^{-1} (\tilde{S}^T \tilde{d}_k - \tilde{c}_k) , \quad k = 1, 2, \ldots, \infty .$$

(3.29)

By using the relation given in (2.19) and orthogonality relation (2.18), the complex constants $\tilde{g}_k$ given by (3.25) and (3.29) can be simplified to

$$\tilde{g}_0 = i \tilde{A}^T \tilde{c}_0 ,$$

$$\tilde{g}_k = - \frac{1}{2k} B^{-1} (i \tilde{c}_k - \tilde{d}_k) .$$

(3.30)

Hence, the general solution for the elliptic hole subjected to arbitrary loading can be written as

$$\tilde{y} = - 2 \text{Im} \left\{ \tilde{A} \ll \log \tilde{\zeta}_o \gg \tilde{A}^T \right\} \tilde{c}_0 - \sum_{k=1}^{\infty} \frac{1}{k} \text{Re} \left\{ \tilde{A} \ll \zeta_o^{-k} \gg \tilde{g}_k B^{-1} (i \tilde{c}_k - \tilde{d}_k) \right\} ,$$

$$\tilde{\varphi} = - 2 \text{Im} \left\{ \tilde{B} \ll \log \tilde{\zeta}_o \gg \tilde{A}^T \right\} \tilde{c}_0 - \sum_{k=1}^{\infty} \frac{1}{k} \text{Re} \left\{ \tilde{B} \ll \zeta_o^{-k} \gg \tilde{g}_k B^{-1} (i \tilde{c}_k - \tilde{d}_k) \right\} ,$$

(3.31)
where \( \zeta_0, \zeta_k, d_k \) have been given in (3.23b). Applying the stress function given in (3.31) and using a procedure similar to (3.13)-(3.16), we obtain the hoop stress \( \sigma_{nn} \) in real form as

\[
\sigma_{nn} = n^T(\theta) \zeta_n,
\]

\[
\rho \ddot{\zeta}_n = -N_1^T(\theta) \zeta_0 + \sum_{k=1}^{\infty} \{ [\cos k\psi \tilde{N}_3(\theta) + \sin k\psi \tilde{N}_3(\theta)] L^{-1} \zeta_k
\]

\[
+ [\sin k\psi \tilde{N}_3(\theta) - \cos k\psi \tilde{N}_3(\theta)] L^{-1} d_k \}
\]

where (2.29)_3 and (2.31)_3 are used and

\[
\tilde{N}_3(\theta) = N_3(\theta) \tilde{s} - N_1^T(\theta) \tilde{L}.
\]

The real form solution given above do not contain the eigenvalues \( p_\alpha \) and the eigenvectors \( \zeta_\alpha, \tilde{\zeta}_\alpha \). Therefore, it avoids the problem of repeated eigenvalues and can be applied to any degenerated materials such as isotropic materials.

### 3.2.1 Uniform Loading

Consider the plate with a traction-free hole subjected to a uniform loading at infinity. Due to the linear property, the principle of superposition can be used and the solution for this problem can be represented by the sum of an unnotched plate and corrective solution, for which the loading applied on the boundary are

\[
\ddot{\zeta}_m = -\overset{*}{T} \overset{*}{N}(\theta),
\]

where

\[
\overset{*}{T} = \begin{bmatrix} t_1^\infty & t_2^\infty & t_3^\infty \end{bmatrix},
\]

and

\[
\begin{align*}
\overset{\infty}{t}_1^1 &= \begin{bmatrix} \sigma_{11}^{\infty} \\ \sigma_{12}^{\infty} \\ \sigma_{13}^{\infty} \end{bmatrix}, & \overset{\infty}{t}_2^1 &= \begin{bmatrix} \sigma_{21}^{\infty} \\ \sigma_{22}^{\infty} \\ \sigma_{23}^{\infty} \end{bmatrix}, & \overset{\infty}{t}_3^1 &= \begin{bmatrix} \sigma_{31}^{\infty} \\ \sigma_{32}^{\infty} \\ \sigma_{33}^{\infty} \end{bmatrix}
\end{align*}
\]
\(\sigma_{ij}^\infty\) are constant stresses applied at infinity. By using the relation of \(\theta\) and \(\psi\) shown in (3.14), substitution of (3.34) into (3.23b) gives the coefficients of Fourier expansion as

\[
\zeta_0 = \zeta_k = d_k = 0, \quad k \neq 1
\]
\[
\zeta_1 = -b \zeta_1^\infty, \quad d_1 = -a \zeta_2^\infty.
\]  
(3.35)

The displacements, stress functions and the hoop stress given in (3.31) and (3.32) will then become

\[
y = \text{Re} \left\{ A \ll \log \zeta_0 \gg B^{-1}(ib \zeta_1^\infty - a \zeta_2^\infty) \right\},
\]
\[
\phi = \text{Re} \left\{ B \ll \zeta_0^{-k} \gg B^{-1}(ib \zeta_1 - a \zeta_2) \right\},
\]
\[
\sigma_{nn} = \zeta_3^T(\theta) \left\{ [\sin \theta \zeta_3^T(\theta) - \frac{b}{a} \cos \theta \zeta_3^T(\theta)] \zeta_1^\infty
\right.
\]
\[
- [\cos \theta \zeta_3^T(\theta) + \frac{a}{b} \sin \theta \zeta_3^T(\theta)] \zeta_2^\infty \right\},
\]  
(3.36)

which are equivalent to those shown in Section 3.1.2.

3.2.2 Pure bending

In this case, we consider the plate subjected to pure bending \(M\) at infinity in the direction at an angle \(\alpha\) with positive \(x_1\)-axis. By superposition with an unnotched plate, the hole boundary is subjected to the following equilibrated stress state,

\[
\sigma_{11} = -\frac{M}{I} (x_1 \sin \alpha - x_2 \cos \alpha) \cos \alpha n_1(\alpha)
\]
\[
\sigma_{12} = -\frac{M}{I} (x_1 \sin \alpha - x_2 \cos \alpha) \sin \alpha n_1(\alpha).
\]  
(3.37a)

The arbitrary loading \(\zeta_m\) considered in (3.31) is expressed in the form of surface traction along the hole boundary of which the normal is \(m\), i.e., \(\zeta_m = \sigma_{ij} m_j\). Hence

\[
\zeta_m = -\zeta^* m
\]  
(3.38a)

where

\[
\zeta^* = -\frac{M}{I} (a \cos \psi \sin \alpha - b \sin \psi \cos \alpha) \zeta^T(\alpha).
\]  
(3.38b)
Substituting (3.38) into (3.23b), we have

\[ \zeta_0 = \zeta_k = \zeta_k = \zeta_2 = 0, \quad k \neq 2 \]
\[ \zeta_2 = \frac{abM}{2I} \sin 2\alpha \hat{\eta}(\alpha) \]
\[ d_2 = \frac{M}{2I}(a^2 \sin^2 \alpha - b^2 \cos \alpha) \hat{\eta}(\alpha) \quad (3.39) \]

By (3.31), (3.32), and (3.39), the displacements, stress functions and hoop stress become

\[ y = \frac{-M}{4I} \text{Re} \left\{ (iab \sin 2\alpha - a^2 \sin^2 \alpha + b^2 \cos^2 \alpha) A \ll \zeta^{-2} \gg B^{-1} \right\} \hat{\eta}(\alpha) \]
\[ \phi = \frac{-M}{4I} \text{Re} \left\{ (iab \sin 2\alpha - a^2 \sin^2 \alpha + b^2 \cos^2 \alpha) B \ll \zeta^{-2} \gg B^{-1} \right\} \hat{\eta}(\alpha) \quad (3.40) \]
\[ \sigma_{nn} = \frac{M}{2pl} \right( \frac{\hat{\eta}}{2} T(\theta) \right) \left\{ ab \sin 2\alpha [\cos 2\psi \hat{N}_3(\theta) + \sin 2\psi \hat{N}_3(\theta)] + (a^2 \sin^2 \alpha - b^2 \cos^2 \alpha) [\sin 2\psi \hat{N}_3(\theta) - \cos 2\psi \hat{N}_3(\theta)] \right\} \hat{\eta}(\alpha) \]

which are equivalent to those shown in (Hwu, 1990).

### 3.2.3 Concentrated loading

Consider a concentrated force \( \hat{p} \) acting at the point \( x_1 = a \cos \psi_0, \ x_2 = b \sin \psi_0 \) on the hole boundary. By the force equilibrium equation

\[ -\int_{-\pi}^{\pi} \hat{t}_m \rho d\psi = \hat{p} \quad , \quad (3.41) \]

the surface loading \( \hat{t}_m \) can be represented by

\[ \hat{t}_m = -\hat{p} \delta(\psi - \psi_0) \quad , \quad (3.42) \]

where \( \delta(\psi - \psi_0) \) is Delta function with an impluse located at \( \psi = \psi_0 \). Substituting (3.42) into (3.23b), we obtain

\[ \zeta_0 = -\frac{1}{2\pi} \hat{p}, \quad \zeta_k = -\frac{1}{\pi} \hat{p} \cos k\psi_0, \]
\[ d_k = -\frac{1}{\pi} \hat{p} \sin k\psi_0 \quad . \quad (3.43) \]
The displacements, stress functions and hoop stress given in (3.31) and (3.32) will then become

\[
\begin{align*}
\tilde{u} &= \frac{1}{\pi} \text{Im} \left\{ \tilde{A} \ll \log \zeta \gg \tilde{A}^T \right\} \tilde{\rho} + \sum_{k=1}^{\infty} \frac{1}{\pi k} \text{Re} \left\{ i e^{i k \psi_0} \tilde{A} \ll \zeta^{-k} \gg B^{-1} \right\} \tilde{\rho} \\
\tilde{\phi} &= \frac{1}{\pi} \text{Im} \left\{ \tilde{B} \ll \log \zeta \gg \tilde{A}^T \right\} \tilde{\rho} + \sum_{k=1}^{\infty} \frac{1}{\pi k} \text{Re} \left\{ i e^{i k \psi_0} \tilde{B} \ll \zeta^{-k} \gg B^{-1} \right\} \tilde{\rho} \\
\sigma_{nn} &= \frac{1}{\pi \rho \eta^T} \left( \frac{1}{2} N_1^T(\theta) - \sum_{k=1}^{\infty} \left[ \cos k(\psi - \psi_0) \tilde{N}_3(\theta) + \sin k(\psi - \psi_0) \tilde{N}_1(\theta) \right] \right) \tilde{\rho}^{-1} \\
&= \frac{1}{\pi} \text{Im} \left\{ \tilde{A} \ll \log(\zeta - e^{i \psi_0}) \gg \tilde{A}^T \right\} \tilde{\rho} \\
&+ \frac{1}{\pi} \text{Im} \left\{ \tilde{B} \ll \log(\zeta^{-1} - e^{-i \psi_0}) \gg \tilde{B}^{-1} \tilde{A}^T \right\} \tilde{\rho}
\end{align*}
\]

Knowing that

\[
- \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{e^{i \psi_0}}{\zeta} \right)^k = \log(1 - \frac{e^{i \psi_0}}{\zeta}) \quad ,
\]

where \( |\zeta| > 1 \), the first two equations of (3.44) can be proved to be equivalent to those shown in Section 3.1.1, i.e.,

\[
\begin{align*}
\tilde{u} &= \frac{1}{\pi} \text{Im} \left\{ \tilde{A} \ll \log(\zeta - e^{i \psi_0}) \gg \tilde{A}^T \right\} \tilde{\rho} \\
&+ \frac{1}{\pi} \text{Im} \left\{ \tilde{A} \ll \log(\zeta^{-1} - e^{-i \psi_0}) \gg \tilde{B}^{-1} \tilde{A}^T \right\} \tilde{\rho} \\
\tilde{\phi} &= \frac{1}{\pi} \text{Im} \left\{ \tilde{B} \ll \log(\zeta - e^{i \psi_0}) \gg \tilde{A}^T \right\} \tilde{\rho} \\
&+ \frac{1}{\pi} \text{Im} \left\{ \tilde{B} \ll \log(\zeta^{-1} - e^{-i \psi_0}) \gg \tilde{B}^{-1} \tilde{A}^T \right\} \tilde{\rho} \quad ,
\end{align*}
\]
CHAPTER IV
FINITE BODIES WITH HOLES

The problems of finite bodies with an elliptic hole are solved by using boundary element method. It is well known that the boundary element method need fundamental solutions which have been presented in section 3.1.1. Since the fundamental solution satisfies the hole boundary condition \textit{a priori}, the integration along the elliptic boundary can be avoided. Several examples are considered to verify the accuracy and efficiency of the present boundary element method.

4.1 Boundary Element Method

\textit{Boundary integral equation}

If body forces are omitted, the boundary integral equation is written as \cite{Brebbia et al., 1984}

\[ c_{ij}(\hat{x})u_j(\hat{x}) + \int_{B+L} \hat{T}_{ij}(\hat{x}, \hat{z})u_j(\hat{z})d\Gamma(\hat{z}) = \int_{B+L} \hat{U}_{ij}(\hat{x}, \hat{z})t_j(\hat{z})d\Gamma(\hat{z}) , \]  

where \( \hat{U}_{ij}(\hat{x}, \hat{z}) \) and \( \hat{T}_{ij}(\hat{x}, \hat{z}) \) are, respectively, the displacements and tractions in the \( j \) direction at point \( \hat{z} = (x_1, x_2) \) corresponding to a unit point force acting in the \( i \) direction applied at point \( \hat{x} = (\hat{x}_1, \hat{x}_2) \) and \( B, L \) denote the contour of the outer and hole boundary. \( c_{ij}(\hat{x}) \) is a coefficient to be determined by the boundary geometry. For a smooth boundary \( c_{ij} = \frac{1}{2} \delta_{ij} \), in which \( \delta_{ij} \) is the Kronecker delta. In general, the value of \( c_{ij}(\hat{x}) \) can be estimated by considering the rigid body translation.

For the traction-free hole problem, we choose \( \hat{U}_{ij} \) and \( \hat{T}_{ij} \) in such a way that \( \hat{T}_{ij} = 0 \) on the elliptic boundary \( L \). Then, by applying the boundary condition \( t_j = 0 \) on \( L \), we find that

\[ \int_{L} \hat{T}_{ij}(\hat{x}, \hat{z})u_j(\hat{z})d\Gamma(\hat{z}) = \int_{L} \hat{U}_{ij}(\hat{x}, \hat{z})t_j(\hat{z})d\Gamma(\hat{z}) = 0 . \]  

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Substituting (3.7) and (3.18) into (3.3) and using (2.12), \( \widehat{\mathbf{U}} = [\widehat{U}_{ij}] \), \( \widehat{\mathbf{T}} = [\widehat{T}_{ij}] \) are obtained as

\[
\widehat{\mathbf{U}} = \frac{1}{\pi} \text{Im} \left\{ \mathbf{A} \ll \log(\zeta_\alpha - \zeta_\alpha) \gg \mathbf{A}^T \right\} \\
+ \frac{1}{\pi} \sum_{k=1}^{3} \text{Im} \left\{ \mathbf{A} \ll \log(\zeta_k^{-1} - \zeta_k) \gg \mathbf{B}^{-1} \overline{\mathbf{B}} I_k \mathbf{A}^T \right\}
\]

\[
\widehat{\mathbf{T}} = \frac{1}{\pi} \text{Im} \left\{ \mathbf{B} \ll \frac{2\zeta_0^2(s_1 + p_0 s_2)}{(\zeta_0 - \zeta_0)(a - ip_\alpha b)\zeta_0^2 - (a + ip_\alpha b)} \gg \mathbf{A}^T \right\} \\
+ \frac{1}{\pi} \sum_{k=1}^{3} \text{Im} \left\{ \mathbf{B} \ll \frac{-2\zeta_0(s_1 + p_0 s_2)}{(1 - \zeta_0 \zeta_k)(a - ip_\alpha b)\zeta_0^2 - (a + ip_\alpha b)} \gg \mathbf{B}^{-1} \overline{\mathbf{B}} I_k \mathbf{A}^T \right\}
\]

(4.3)

where \( s_1 = \frac{\partial \xi_1}{\partial s} \), \( s_2 = \frac{\partial \xi_2}{\partial s} \).

If \( \tilde{\zeta} \) is an internal point, \( c_{ij} \) in (4.1) becomes \( \delta_{ij} \). Equation (4.1) is a continuous representation of displacement at any point \( \tilde{\zeta} \) inside body. Differentiating (4.39) with respect to \( \tilde{\zeta} \) and then substituting its result into strain-displacement relation (2.1) and stress-strain laws (2.2), the internal stresses at point \( \tilde{\zeta} \) can be written as

\[
\sigma_1(\tilde{\zeta}) = \int_B \left( R \frac{\partial \widehat{U}_1(\tilde{\zeta}, \zeta)}{\partial \tilde{\xi}_1} + T \frac{\partial \widehat{U}_1(\tilde{\zeta}, \zeta)}{\partial \tilde{\xi}_2} \right) \tau(\zeta) d\Gamma(\zeta)
- \int_B \left( R \frac{\partial \widehat{U}(\tilde{\zeta}, \zeta)}{\partial \tilde{\xi}_1} + T \frac{\partial \widehat{U}(\tilde{\zeta}, \zeta)}{\partial \tilde{\xi}_2} \right) \tau(\zeta) d\Gamma(\zeta),
\]

(4.4a)

\[
\sigma_2(\tilde{\zeta}) = \int_B \left( R \frac{\partial \widehat{U}_2(\tilde{\zeta}, \zeta)}{\partial \tilde{\xi}_1} + T \frac{\partial \widehat{U}_2(\tilde{\zeta}, \zeta)}{\partial \tilde{\xi}_2} \right) \tau(\zeta) d\Gamma(\zeta)
- \int_B \left( R \frac{\partial \widehat{U}(\tilde{\zeta}, \zeta)}{\partial \tilde{\xi}_1} + T \frac{\partial \widehat{U}(\tilde{\zeta}, \zeta)}{\partial \tilde{\xi}_2} \right) \tau(\zeta) d\Gamma(\zeta),
\]

where

\[
\sigma_1 = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{bmatrix},
\]

(4.4b)

and

\[
\tau = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.
\]

(4.4c)
Numerical procedure

The boundary integral equation (4.1) must, in general, be solved numerically. Here linear elements are used, so that the values of \( \tilde{u} \) and \( \tilde{t} \) over each element can be expressed in terms of nodal displacements \( \tilde{u}_e \) and traction \( \tilde{t}_e \) as

\[
\tilde{u} = \tilde{\varpi} \tilde{u}_e, \quad \tilde{t} = \tilde{\varpi} \tilde{t}_e, \tag{4.5a}
\]

where

\[
\tilde{\varpi} = [\tilde{\varpi}_1, \tilde{\varpi}_2]_{3 \times 6}, \quad \tilde{\varpi}_i = \tilde{\varpi}_i \tilde{I}_e \tag{4.5b}
\]

\[
\tilde{u}_e = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \tilde{t}_e = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}, \tag{4.5c}
\]

\[
\tilde{\varpi}_i = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \tilde{t}_i = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}, \tag{4.5d}
\]

and the shape functions \( \tilde{\varpi}_i \) are

\[
\tilde{\varpi}_1 = \frac{1}{2} (1 - \eta), \tag{4.5e}
\]

\[
\tilde{\varpi}_2 = \frac{1}{2} (1 + \eta).
\]

If the boundary \( B \) is discretized into \( M \) segments with \( N \) nodes, substitution (4.5) into (4.1) with (4.2) yields the approximation

\[
\mathcal{C} \tilde{u} + \sum_{i=1}^{M} \int_{\Gamma_i} \mathcal{P} \tilde{\varpi} d\Gamma_i \tilde{u}_e = \sum_{i=1}^{M} \int_{\Gamma_i} \mathcal{U} \tilde{\varpi} d\Gamma_i \tilde{t}_e, \tag{4.6}
\]

where \( \Gamma_i \) denotes the \( i \)th segment of the discretized boundary. With use of (4.5b), equation (4.6) which gives a set of equation for a given node of source points becomes

\[
\mathcal{C} \tilde{u} + \sum_{i=1}^{M} \mathcal{H} \tilde{u}_e = \sum_{i=1}^{M} \mathcal{G} \tilde{t}_e, \tag{4.7a}
\]

where

\[
\mathcal{H} = \int_{\Gamma_i} \begin{bmatrix} \tilde{\varpi}_1 & \tilde{\varpi}_2 \end{bmatrix} d\Gamma_i, \tag{4.7b}
\]

\[
\mathcal{G} = \int_{\Gamma_i} \begin{bmatrix} \tilde{\varpi}_1 & \tilde{\varpi}_2 \end{bmatrix} d\Gamma_i.
\]
In (4.7), for each position of the source point \( \hat{z} \) one obtains three algebraic equations. As the source point passes through all \( N \) introduced nodal points, one can obtain \( 3N \) linear algebraic equations. By applying the boundary condition where either \( u_i \) or \( t_i \) at each node is prescribed, a system of \( 3N \) simultaneous linear algebraic equations with \( 3N \) unknown is obtained and the unknown displacements and tractions along the boundary can be found by solving this system of equations. The coefficients of \( \zeta \) together with the corresponding principal value of \( \hat{H} \) can be indirectly computed by consideration of rigid-body movements in the body i.e., by letting \( u_j = 1 \) (and hence \( p_j = 0 \)) in Eq.(4.7). Each corner node connects two different surfaces which have two different tractions, it should be represented by two nodes.

The integrals including the one with integral terms such as \( \log(\zeta_\alpha - \hat{\zeta}_\alpha) \) and \( 1/(\zeta_\alpha - \hat{\zeta}_\alpha) \) in (4.3) are evaluated by using Gauss quadrature rules. In this study the required Gauss points for integrating (4.7b) depend on the distance between source point and the midpoint of each element because the smaller the distance, the larger the variation of tractions and displacements. When the source point is located on the element 16 Gauss points are needed due to existence of singular integral.

Once all the values of tractions and displacements on the boundary are determined, the values of stresses and displacements at any interior point can be obtained by numerical integral of (4.4a).

4.2 Examples

(1) A finite plate or an infinite plate with an elliptic hole

A uniform tension of \( \sigma_0 = 1GPa \) is applied in the \( x_2 \)-direction. The material constants
of the orthotropic plate are taken as

\[ E_1 = 11.8 \text{GPa} \quad , \quad E_2 = 5.9 \text{GPa} \quad , \]
\[ G_{12} = 0.69 \text{GPa} \quad , \quad \nu_{12} = 0.071 \quad , \]

where fiber direction is denoted by 1. Lamina angle \( \alpha \) is 90°, where \( \alpha \) is measured from \( x_1 \)-axis to fiber direction. Because of Eq. \((4.2)\), only the external boundary need to be discretized. An infinite plate containing a circular hole is simulated by defining \( b/a = 1, \quad 2a/W = 0.01, \quad H/W = 3 \) (Figure 2). A very coarse mesh is employed in this case with four elements on each edge. Therefore, twenty nodes for the entire plate are required. Results for the hoop stress \( \sigma_{\theta \theta} \) are shown in Table 1. We observed that there is a very good agreement between the analytic (Lekhnitskii, 1968) and computed solution by the boundary element method.

The main concern of the following cases is stress concentration factor \((SCF)\) which is defined by \( \sigma_{\theta \theta}/\sigma_0 \) at point A. The \( SCF \) is presented in Figure 3 for a variety of \( b/a \) ratios in order to investigate the effect of the shape of elliptic hole on the stress concentration factor, where \( 2a/W = 0.2, H/W = 3, \alpha = 90^\circ \). The results of Figure 3 show that the larger the \( b/a \) ratio, the smaller the stress concentration factor, which is expected. The relation between \( SCF \) and lamina angle \( \alpha \) is shown in Figure 4.

\((2)\) A finite plate with a pin-loaded hole

A finite plate with a hole is shown in Figure 5, which is subjected to a uniform tension applied at \( x_2 = -H_2 \). The material constants of the orthotropic plate are taken as

\[ E_1 = 10 \text{GPa} \quad , \quad E_2 = 10 \text{GPa} \quad , \]
\[ G_{12} = 10/3.3 \text{GPa} \quad , \quad \nu_{12} = 0.25 \quad . \]

A cosine normal load distribution is assumed here to simulate a pin-loaded hole (Chang \textit{et al.}, 1983; Vable and Sikarskie, 1988). The pressure around the upper half of the hole
boundary induced by the unit stress applied in the \( x_2 \)-direction can then be expressed as

\[
\sigma_{mm} = P \cos \theta^* ,
\]

where \( \theta^* \) is measured clockwise from \( x_2 \)-axis, \( \mathbf{n} \) is the direction normal to hole boundary, \( P \) is determined by the equilibrium condition that the total forces in the vertical direction balance, i.e.,

\[
2 \int_0^{\pi/2} (P \cos^2 \theta^*) \frac{d \theta^*}{2} = \frac{\pi}{4} Pd = 1 \times W .
\]

From the above equation, \( P \) is given by \( 4W/\pi d \). The fundamental solution derived in section 3.1.1 is valid under the condition that the hole is free of traction. In order to solve the problem with load distribution described above, superposition of the following two problems is employed. One is an infinite plate with \( \sigma_{mm} = \frac{4W}{\pi d} \cos \theta^* \) applied on the upper half of the hole boundary, which can be solved analytically by using the results of section 3.2, the other is a finite plate (Figure 5) of which the loadings applied on the outer and hole boundaries is obtained from those of the original problem minus the above one. Therefore, the loading on the hole boundary is traction free so that it can be solved by the present boundary element method. The results for the first part of the solution are as follows. The prescribed surface traction \( \tilde{t}_m \) for the pressure \( \sigma_{mm} = P \cos \theta^* \) can be described as

\[
\tilde{t}_m = P \cos \theta^* \begin{pmatrix}
\sin \theta^* \\
\cos \theta^* \\
0
\end{pmatrix} = P \begin{pmatrix}
\sin \psi \cos \psi \\
\sin^2 \psi \\
0
\end{pmatrix} , \quad 0 \leq \psi \leq \pi ,
\]

\[
= \tilde{0} , \quad \pi \leq \psi \leq 2\pi . \quad (4.8)
\]

From (3.23b) and (4.8) with \( a = b = d/2 \), we have

\[
\zeta_0 = \frac{dP}{4\pi} \begin{pmatrix}
0 \\
\pi/2 \\
0
\end{pmatrix} ,
\]

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\[ \tilde{c}_k = \frac{dP}{4\pi} \begin{cases} \frac{4}{4-k^2} & \text{, } k = \text{odd}, \\ 0 & \text{, } k = \text{even except 2}, \end{cases} \]

\[ = \tilde{\circ} \quad , \quad k = \text{even except 2}, \]

\[ = \frac{dP}{4\pi} \begin{cases} 0 & \text{, } k = 2, \end{cases} \]

\[ \tilde{d}_k = \frac{dP}{4\pi} \begin{cases} 0 & \text{, } k = \text{odd}, \end{cases} \]

\[ = \tilde{\circ} \quad , \quad k = \text{even except 2}, \]

\[ = \frac{dP}{4\pi} \begin{cases} \pi/2 & \text{, } k = 2. \end{cases} \]

(4.9)

The displacements and stresses for the entire plate containing a pin-loaded hole can then be calculated by substituting (4.9) into (3.31) and (2.10).

The final results combining these two parts for normal stress \( \sigma_{22} \) along \( x_1 \)-axis are shown and compared in Figure 6. Results for the total force along the \( x_1 \)-axis are shown in Table 2.

(3) Convergency and accuracy.

A sample problem dealing with a traction-free hole under uniaxial tension given in (Vable and Sikarskie, 1988), Figure 5, is used to study the convergency and accuracy of the present method. A uniform tension is applied at \( x_2 = H_1 \) and \(-H_2\). All the other boundaries including the circular hole boundary are traction free and the material properties are the same as the above Example. The effect of varying \( N \) (number of nodes) on the stress concentration factor is shown in Table 3. Good convergence rate is observed from \( N=8 \) to \( N=42 \). Results for normal stress \( \sigma_{22} \) along positive \( x_1 \)-axis are shown in Figure 6. By static equilibrium the area under the curve shown in Figure 6 represents half the total force applied in the \( x_2 \)-direction, which is shown in Table 2.
CHAPTER V
CRACK PROBLEMS

5.1 Stress intensity factors

In the following, the cases of point forces are studied. The stress function \( \hat{\sigma} \) for the crack problems can be found by substituting (3.7) and (3.18) into (3.3), and letting the minor axis of ellipse approach to zero. Differentiating stress function with respect to \( x_1 \) and considering \( x_2 = 0, x_1 > a \), the stresses \( \sigma_{i2} \) ahead of the crack tip along \( x_1 \) axis are obtained as

\[
\sigma_2 = \frac{1}{\pi a} \left( 1 + \frac{x_1}{\sqrt{x_1^2 - a^2}} \right) \text{Im} \left\{ \sum_{\alpha=1}^{3} \left( \frac{1}{\zeta - \zeta_\alpha} + \frac{1}{\zeta - \zeta^2 \zeta_\alpha} \right) \dot{b}_\alpha \dot{r}_\alpha^T \right\} \hat{p}, \quad (5.1a)
\]

where

\[
\zeta = \frac{1}{a} (x_1 + \sqrt{x_1^2 - a^2}) \quad , \quad \dot{\zeta}_\alpha = \frac{1}{a} (\dot{x}_\alpha + \sqrt{\dot{x}_\alpha^2 - a^2}) \quad , \quad (5.1b)
\]

and

\[
\sigma_2 = \{\sigma_{21}, \sigma_{22}, \sigma_{23}\}^T \quad . \quad (5.1c)
\]

With the usual definition

\[
\tilde{K} = \left\{ \begin{array}{c} K_{II} \\ K_I \\ K_{III} \end{array} \right\} = \lim_{x_1 \to a} \sqrt{2\pi(x_1 - a)\sigma_2}, \quad (5.2)
\]

and use of (5.1a), the stress intensity factors \( \tilde{K} \) are given by

\[
\tilde{K} = \frac{2}{\sqrt{\pi a}} \text{Im} \{ B \ll \frac{1}{1 - \zeta_\alpha} \gg A^T \} \hat{p}. \quad (5.3)
\]

If the point force \( \hat{p} \) is applied on the upper crack surface \( x_1 = c \), equation (5.3) becomes

\[
\tilde{K} = \frac{-1}{2\sqrt{\pi a}} S^T \hat{p} + \frac{1}{2\sqrt{\pi a}} \left( \frac{a + c}{a - c} \right)^{1/2} \hat{p} \quad , \quad (5.4)
\]

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where the identity (2.19) has been used and $K$ is identical to the solution given by Wu (1989).

For finite plates, the stress intensity factors can be calculated by substituting (4.4) into (5.2) and utilizing

$$\tilde{\zeta}_k = \hat{\zeta}_k = \hat{\zeta} = \frac{1}{a}(\hat{x}_1 + \sqrt{\hat{x}_1^2 - a^2})$$

$$\zeta_\alpha = \frac{1}{a}(z_\alpha + \sqrt{z_\alpha^2 - a^2})$$

along $\hat{x}_2 = 0$,

$$\lim_{\hat{x}_1 \to 0} \hat{\zeta} = 1,$$

$$\lim_{\hat{x}_1 \to 0} \sqrt{2\pi(\hat{x}_1 - a)} \frac{1}{\zeta^2 - 1} = \frac{\sqrt{\pi a}}{2}.$$

The results are

$$K = \int_B (R_1^T \Theta_T + T_1 \Theta_{II}) \xi \, d\Gamma(\xi) - \int_B (R_1^T \Lambda_T + T_1 \Lambda_{II}) \xi \, d\Gamma(\xi)$$

(5.5a)

where

$$\Theta_I = \frac{1}{\sqrt{\pi a}} \text{Im} \left\{ A \ll \frac{1}{1 - \zeta_\alpha} \gg A^T \right\}$$

$$+ \frac{1}{\sqrt{\pi a}} \sum_{k=1}^3 \text{Im} \left\{ A \ll \frac{-\zeta_\alpha}{1 - \zeta_\alpha} \gg B^{-1} \overline{B} I_k \overline{A}^T \right\}$$

$$\Theta_{II} = \frac{1}{\sqrt{\pi a}} \text{Im} \left\{ A \ll \frac{p_\alpha}{1 - \zeta_\alpha} \gg A^T \right\}$$

$$+ \frac{1}{\sqrt{\pi a}} \sum_{k=1}^3 \text{Im} \left\{ A \ll \frac{-p_k \zeta_\alpha}{1 - \zeta_\alpha} \gg B^{-1} \overline{B} I_k \overline{A}^T \right\}$$

$$\Lambda_T = \frac{2}{a \sqrt{\pi a}} \text{Im} \left\{ B \ll \frac{(s_1 + p_\alpha s_2)}{\zeta_\alpha^2 - 1} \gg A^T \right\}$$

$$- \frac{1}{a \sqrt{\pi a}} \sum_{k=1}^3 \text{Im} \left\{ B \ll \frac{(s_1 + p_\alpha s_2)}{(1 - \zeta_\alpha)^2(\zeta_\alpha^2 - 1)} \gg B^{-1} \overline{B} I_k \overline{A}^T \right\}$$

(5.5b)

$$\Lambda_{II} = \frac{2}{a \sqrt{\pi a}} \text{Im} \left\{ B \ll \frac{p_\alpha(s_1 + p_\alpha s_2)}{\zeta_\alpha^2 - 1} \gg A^T \right\}$$

$$- \frac{1}{a \sqrt{\pi a}} \sum_{k=1}^3 \text{Im} \left\{ B \ll \frac{-p_k(s_1 + p_\alpha s_2)}{(1 - \zeta_\alpha)^2(\zeta_\alpha^2 - 1)} \gg B^{-1} \overline{B} I_k \overline{A}^T \right\}$$

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Equation (5.5) provides a direct method for evaluating the stress intensity factor if remote tractions and displacements have been known on some closed contour containing the crack. It should be noted that the data on the closed contour may be supplied by any method including finite element.

5.2 Examples

(1) A composite laminate with a center-crack under uniform tension

The geometry and loading for this problem are shown in Figure 2 with $b \to 0^+$, $H/W = 3$. Numerical results are obtained for three laminates: $(90^\circ)_s$, $(\pm 30^\circ)_s$, $(0^\circ/\pm 45^\circ/90^\circ)_s$, which are relevant to composite structure application. The material properties of each lamina are denoted by

$$E_1 = 114.8\text{GPa}$$
$$E_2 = E_3 = 11.72\text{GPa}$$
$$G_{12} = G_{13} = G_{23} = 9.65\text{GPa}$$
$$\nu_{12} = \nu_{13} = \nu_{23} = 0.21$$

To study the effects of the specimen boundary the results are presented using the stress-intensity correction factor ($F$)

$$F = \frac{K_I}{\sigma \sqrt{\pi a}}$$

Figure 7 shows that the stress-intensity correction factors calculated by this method agree, within $\pm 1\%$, with the values from Snyder and Cruse (1975).

(2) An orthotropic plate with a center-crack under antiplane loading

A mode III problem is now considered to show the generality of this method. An infinite plate is simulated by defining $2a/W = 0.01$ in Figure 8. The plate is sheared by a stress $\sigma_{23} = \tau$ on the $x_2 = \pm H/2$. For orthotropic materials, we take the lamina properties considered in the above example and orient the fiber direction to $x_3$-axis. The
stress intensity factor \( K_{II} \) calculated converges to \( 1.001 \tau \sqrt{\pi a} \), which shows that it is not influenced by material properties for small crack.
CHAPTER VI
CONCLUSIONS

In this research, the laminated composites are modelled by an anisotropic plate. A
general solution for the problems of two-dimensional anisotropic plates containing an el-
liptic hole or crack is formulated by applying the Stroh's formalism. To apply for the
elliptic boundary and arbitrary loading conditions, the technique of conformal mapping
and a special method of analytical continuation are developed. Based upon the choices
of unperturbed stress functions, two different possible conditions are considered. One is
represented by the Taylor's series, the other is represented by Laurent's expansion. The
final solutions will not be affected by the choices of unperturbed stress functions since
two types can all capture singular behavior of loading. Some special loadings are solved
explicitly such as the point forces and uniform loadings. The elasticity solutions of the
point force problems can be used as the fundamental solutions of the boundary element, in
which the discretization along the hole boundary can be avoided. Therefore, less computer
time and storage, and higher accuracy can be achieved. The case of uniform loading is the
one that detailed and complete analysis has been provided in the literature, and is used here
to verify the correctness of our results.
REFERENCES


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Mukherjee, S. and Morjaria, M., 1981a, "A Boundary Element Formulation for Planer


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\[ x_1 = a \cos \psi \]
\[ x_2 = b \sin \psi \]
Figure 6 shows a graph of $\sigma_{22}$ vs. $x_1/d$. The graph compares different conditions:

- The solid line with filled circles represents the 'Present' data.
- The dashed line with open triangles represents 'Vable and Sikarskie (1988)' data.

The graph also indicates two types of holes:

- Pin-loaded hole
- Traction-free hole