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姓名：（中文）胡澄清

（英文）Chyanbin Hwu

職稱：（中文）副教授

（英文）Associate Professor

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The Onset and Growth of Delaminations and Its Applications

Part I

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Chyanbin Hwu
Wen J. Yen and Jian S. Hu

Institute of Aeronautics and Astronautics
National Cheng Kung University
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ABSTRACT

By combining the analytical continuation method and Stroh's formalism, the analytical solutions of the stress intensity factors $K_I, K_{II}, K_{III}$, the total strain energy release rate $G$ and the modified $G_1, G_2, G_3$ for the interfacial crack between two dissimilar anisotropic media are obtained in this report. Based upon this result, the dependence of stress intensity factors and energy release rates on the fiber orientations is studied analytically. Simple curve-fitted functions are obtained for the relations between these fracture parameters and ply orientations, which may be useful for the future experimental work. In order to save the computer time and storage, a finite 3D elasticity problem is reduced to two 2D problems, i.e., the interior and boundary layer problems, by the asymptotic analysis, which has been formulated in this report. However, the programming and testing have not been finished, which are expected to be accomplished in the next stage of this research.
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CHAPTER 1
INTRODUCTION

One of the most frequently encountered problems in composite laminates is interface cracking, sometimes also known as delamination. Delaminations in layered composite materials may occur due to a variety of reasons, such as low energy impact, manufacturing defects or high stress concentrations at geometric or material discontinuities (e.g. the well-known free edge effects). However, owing to the intrinsic complexities involved such as anisotropy and inhomogeneity, a satisfactory, physically meaningful and universal delamination failure criterion has not yet been found.

Regardless of the nature of loading, delamination has been thought to be driven by the interlaminar stresses. Early analytical efforts have concentrated, therefore, on the determination of the interlaminar stresses which were subsequently used to predict the onset and growth of delamination, for example [1,2]. Due to the singularity nature of delamination, the fundamental concept of classical fracture mechanics was applied later on. However, a general combination of the fracture modes involving fiber breaking and matrix cracking exhibited by fiber reinforced composites may not satisfy the assumptions of linear elastic fracture mechanics, which require that the material is homogeneous on some size scale and the crack growth is self-similar. Delamination is one of the few cases where fracture mechanics can be applied aptly.

A quantitative assessment of the effect of realistic delaminations on the strength and lifetime of a laminate is difficult. Consequently, analytical efforts to date have only attempted to quantify the effect of idealized delaminations [3-15]. The laminate studies were so designed and loaded as to induce delamination growth without significant interaction with other cracking modes. Furthermore, the delamination was confined to grow uniformly
along the straight edge of the test cupon, so it could be assumed as one-dimensional, self-similar growth. Under this assumption, the kinematics of the crack propagation is much simplified, with the crack front being represented by a point known as the crack tip, and the crack size having the magnitude $a$. This simplification has made a two-dimensional stress field solution possible.

Among several parameters used in fracture mechanics, the strain energy release rate $G$ has been shown [3-15] to be a potentially useful tool for the prediction of delamination onset and growth. However, the dependence of critical strain energy release rate $G_c$ (or $G_{1c}$, $G_{2c}$, $G_{3c}$) on the propagation direction, the local fiber or ply-interface orientation and the ply thickness is still not clear. Although there are several failure criteria used in various crack growth situations, questions surrounding the existence and determination method for the constants ($G_c$, $G_{1c}$, $G_{2c}$ or $G_{3c}$) have not been resolved for cracks propagating in composites. Furthermore, different loading conditions may induce different failure modes. Wilkin [16] suggested that Mode I delamination may be predominantly a static load failure, while mixed-mode delamination may be predominantly a fatigue load failure.

Unlike the homogeneous solids for which the stress intensity factors are in most cases independent of the material constants, the stress intensity factors of interfacial cracks may depend on the material constants of upper and lower media. In other words, the stress intensity factors may be a function of ply orientation in composite laminates, so is the strain energy release rate. To know the dependency on the ply orientation, the analytical studies on the interfacial cracks are necessary. Williams [17], Zak and Williams [18], Erdogan [19,20], England [21], Rice and Sih [22], Piva and Viola [23], and many others have studies the static fracture problem of cracks at the interface of dissimilar isotropic solids. A physically unrealistic overlapping of crack surfaces near the crack tip has been
observed. Similar phenomenon has been found by Clement [24], Gotoh [25], and Willis [26] on a crack between dissimilar anisotropic materials. Recently, Rice [27] reexamined the elastic fracture mechanics concepts for the isotropic interfacial cracks and discussed some possible definitions of stress intensity factors which can characterize the strength near the tip of interfacial crack. By applying the Stroh's formalism [28] and Rice's concept, Wu [29] derived the stress intensity factors and total energy release rate for an interfacial crack lying between two anisotropic elastic materials subjected to remote uniform loading. However, due to the oscillatory characteristics, the limits of $G_1$, $G_2$ and $G_3$ do not exist. To extract valuable information of each failure mode, we change the definition from differential to incremental form and derive $G_1$, $G_2$ and $G_3$ analytically from the field solution.

In this report, the Stroh's formalism is combined with the method of analytical continuation [30] to solve the interface crack problems. General analytical solutions for infinite bodies with interface cracks between two dissimilar anisotropic media have been obtained. Two proper definitions for the stress intensity factors are provided. A quadratic relation between the stress intensity factors and newly defined energy release rates is also obtained. Explicit closed form solutions are derived for the orthotropic bimaterials which may be useful for the cross-ply laminates. Some special cases such as a finite crack subjected to uniform loading, a semi-infinite crack subjected to a point load, and a finite crack subjected to a point load, are solved for illustration. By the principal of superposition, the solution of point load problems may be used to attack all the general problems with the same geometry. Numerical studies for the dependency of the stress intensity factors and energy release rates upon the ply orientation are presented by a series of figures. Simple curve-fitted functions are obtained for the relation between these fracture parameters and ply orientations, which may be useful for the future experimental work.
Finally, in order to save the computer time and storage, a finite 3D elasticity problem is reduced to two 2D problems, i.e., the interior and boundary layer problems, by the asymptotic analysis [31]. The interior problems are treated by the classical lamination theory, and can be solved by the usual 2D finite element or boundary element program. A quasi-3D finite element has been formulated in this report to treat the boundary layer problem. The programming and testing have not been finished in this stage, which are expected to be accomplished in the next stage.
CHAPTER 2

TWO-DIMENSIONAL ANISOTROPIC ELASTICITY

2.1 General Solutions

For two-dimensional anisotropic elasticity, there are two different formulations in the literature. One is Lekhnitskii’s approach [32] of which the starting variables are stresses, the other is Stroh’s formalism [28] whose starting variables are displacements. In this report, we follow Stroh’s formalism and the notation employed in [33,34]. In a fixed rectangular coordinate system $x_i, i = 1, 2, 3$, let $u_i, \sigma_{ij}, \varepsilon_{ij}$ be, respectively, the displacement, stress and strain. The strain-displacement equations, the stress-strain laws, and the equations of equilibrium are

\begin{align*}
\varepsilon_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i}) \quad , \\
\sigma_{ij} &= C_{ijks} \varepsilon_{ks} \quad , \\
\sigma_{ij,k} &= C_{ijks} u_{k,s} = 0 \quad ,
\end{align*}

(2.1) (2.2) (2.3)

where repeated indices imply summation, a comma stands for differentiation and $C_{ijks}$ are the elastic constants which are assumed to be fully symmetric and positive definite.

Assuming that $u_i, i = 1, 2, 3$, depend on $x_1$ and $x_2$ only, the general solution to (2.3) can be written in matrix notation as

\[ \underline{u} = \sum_{\alpha=1}^{6} a_\alpha f_\alpha(z_\alpha), \quad z_\alpha = x_1 + p_\alpha x_2 \quad , \]

(2.4)

in which $f_1, f_2, \cdots$ are arbitrary functions of their arguments and $p_\alpha$ and $a_\alpha$ are the eigenvalues and eigenvectors of the following eigenrelation:

\[ \mathcal{D}(p) \underline{a} = 0, \]

(2.5)

\[ \mathcal{D}(p) = \underline{Q} + p(\underline{R} + \underline{R}^T) + p^2 \underline{T}. \]
In (2.5) the superscript $T$ stands for the transpose and $\tilde{Q}, \tilde{R}, \tilde{T}$ are the $3 \times 3$ real matrices given by

$$Q_{ik} = C_{i1k1}, \quad R_{ik} = C_{i1k2}, \quad T_{ik} = C_{i2k2} \quad (2.6)$$

We see that $\tilde{Q}$ and $\tilde{T}$ are symmetric and positive definite if the strain energy is positive [35]. Equation (2.5) is obtained when we substitute (2.4) into (2.3). Since $p_\alpha$ cannot be real if the strain energy is positive, $p_\alpha$ occurs as three pairs of complex conjugates. We let

$$p_{\alpha+3} = \bar{p}_\alpha, \quad \text{Im}(p_\alpha) > 0, \quad \alpha = 1, 2, 3,$$

where an overbar denotes the complex conjugate and Im stands for the imaginary part.

We then have

$$a_{\alpha+3} = \bar{a}_\alpha, \quad \alpha = 1, 2, 3.$$

For the displacement $\tilde{u}$ to be real, we let

$$f_{\alpha+3} = \bar{f}_\alpha, \quad \alpha = 1, 2, 3,$$

and (2.4) becomes

$$\tilde{u} = 2\text{Re} \left\{ \sum_{\alpha=1}^{3} a_\alpha f_\alpha (z_\alpha) \right\}, \quad (2.7)$$

in which Re stands for the real part.

Introducing the vector

$$b = (\tilde{R}^T + p T) \tilde{a} = -\frac{1}{p} (\tilde{Q} + p \tilde{R}) \tilde{a} \quad (2.8),$$

where the second equality comes from (2.5), the stresses $\sigma_{ij}$ obtained by substituting (2.4) into (2.1) and (2.2) can be written as

$$\sigma_{i1} = -\phi_{i,2}, \quad \sigma_{i2} = \phi_{i,1}, \quad (2.9)$$
where $\phi$ is the stress function vector

$$
\phi = 2\text{Re} \left\{ \sum_{\alpha=1}^{3} b_\alpha f_\alpha(z_\alpha) \right\}, \quad (2.10)
$$

and $b_\alpha$ is related to $f_\alpha$ through (2.8). To use the method of analytical continuation, the general solutions given in (2.7) and (2.10) are now rewritten as

$$
y = A f(x) + \overline{A f(x)},
$$

$$
\phi = B f(x) + \overline{B f(x)},
$$

(2.11a)

where

$$
A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix},
$$

$$
f(x) = [f_1(x), f_2(x), f_3(x)]^T.
$$

(2.11b)

Note that the argument of each component function of $f(x)$ is written as $x = x_1 + px_2$ without referring to the associated eigenvalues $p_\alpha$. Once the solution of $f(x)$ is obtained for a given boundary value problem, a replacement of $x_1, x_2$ or $x_3$ should be made for each component function to calculate field quantities from (2.7) and (2.10).

In many applications including the present one, $f_1, f_2, f_3$ of (2.7) and (2.10) have the same function form

$$
f_\alpha(z_\alpha) = q_\alpha f(z_\alpha), \quad \alpha \text{ not summed,}
$$

where $q_\alpha, \alpha = 1, 2, 3$, are arbitrary complex constants. Hence, the general solutions can also be written as

$$
y = 2\text{Re} \{ A F(Z) q \}, \quad \phi = 2\text{Re} \{ B \overline{F(Z)} q \},
$$

(2.12)

in which $q$ is the $3 \times 1$ matrix whose elements are $q_\alpha, \alpha = 1, 2, 3$, and $F(Z)$ is a diagonal matrices,

$$
F(Z) = \text{diag}[f(z_1), f(z_2), f(z_3)],
$$

(2.13)
More generally, if $\tilde{t}$ is the surface traction at a point on a curve boundary, then

$$\tilde{t} = \partial \tilde{\phi} / \partial s,$$

(2.14)

where $s$ is the arc length measured along the curved boundary in the direction such that, when one faces the direction of increasing $s$, the material is located on the right-hand side.

We see that (2.9) are special cases of (2.14) when the boundary is a plane parallel to the $x_2$-axis or the $x_1$-axis.

### 2.2 The Sextic Formalism of Stroh

The two equations in (2.8) can be written in a standard eigenrelation as

$$N \tilde{\xi} = p \tilde{\xi},$$

(2.15)

$$\tilde{N} = \begin{bmatrix} \tilde{N}_1 & \tilde{N}_2 \\ \tilde{N}_3 & \tilde{N}_1^T \end{bmatrix}, \quad \tilde{\xi} = \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix},$$

(2.16)

$$\tilde{N}_1 = -\tilde{T}^{-1} \tilde{R}^T,$$

$$\tilde{N}_2 = \tilde{T}^{-1} = \tilde{N}_2^T,$$

$$\tilde{N}_3 = R \tilde{T}^{-1} \tilde{R}^T - \tilde{Q} = \tilde{N}_3^T.$$

In view of the symmetry and positive definite of $\tilde{T}$, we see that $\tilde{T}^{-1}$ exists.

If we consider a new coordinate system $\tilde{x}_i$, which is obtained by rotating the coordinates $x_i$ about the $x_3$-axis an angle $\omega$, we have

$$\tilde{x}^* = \tilde{\Omega} \tilde{x},$$

(2.18)

$$\tilde{\Omega}(\omega) = \begin{bmatrix} \cos \omega & \sin \omega & 0 \\ -\sin \omega & \cos \omega & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
The elastic constants $C_{ijks}^*$ referred to the new coordinate $x_i^*$ are

$$C_{ijks}^* = \Omega_{ipt} \Omega_{jq} \Omega_{kr} \Omega_{st} C_{pqrt}. \quad (2.20)$$

All the other matrices mentioned in Section 2.1 are then modified as,

$$Q_{ik}^*(\omega) = C_{i1k1}^*, \quad R_{ik}^*(\omega) = C_{i1k2}^*, \quad T_{ik}^*(\omega) = C_{i2k2}^*, \quad (2.21)$$

$$N^*(\omega) \vec{\xi}^*(\omega) = p(\omega) \vec{\xi}^*(\omega), \quad (2.22)$$

$$N^*(\omega) = \begin{bmatrix} N_1^*(\omega) & N_2^*(\omega) \\ N_3^*(\omega) & N_1^*(\omega) \end{bmatrix}, \quad \vec{\xi}^*(\omega) = \Omega(\omega) \begin{bmatrix} a \\ b \end{bmatrix}, \quad (2.23)$$

$$N_1^*(\omega) = -T^{*-1}(\omega) R^* T(\omega), \quad (2.24)$$

$$N_2^*(\omega) = T^{*-1}(\omega) = N_2^* T(\omega),$$

$$N_3^*(\omega) = R^*(\omega) T^{*-1}(\omega) R^* T(\omega) - Q^*(\omega) = N_3^* T(\omega)$$

It has been shown in [36] that $N_1^*(\omega)$ and $N_3^*(\omega)$ have the structures,

$$-N_1^*(\omega) = \begin{bmatrix} r_6(\omega) & 1 & s_6(\omega) \\ r_2(\omega) & 0 & s_2(\omega) \\ r_4(\omega) & 0 & s_4(\omega) \end{bmatrix}, \quad (2.25)$$

$$-N_3^*(\omega) = \begin{bmatrix} e_1(\omega) & 0 & e_2(\omega) \\ 0 & 0 & e_3(\omega) \end{bmatrix}, \quad (2.26)$$

in which $e_i(\omega)$ (i=1,2,3) and $r_i(\omega), s_i(\omega), (i = 2, 4, 6)$ can be expressed in terms of the elastic compliances referred to the $x_i^*$ coordinates. For isotropic materials, $e_i, r_i, s_i$ are independent of $\omega$ and $e_2 = r_4 = r_6 = s_2 = s_4 = s_6 = 0$.

Instead of the above two formalisms which are derived from

$$\ddot{u} = \ddot{a} f(x_1 + px_2), \quad \text{with material constants } C_{ijks}, \quad (2.27)$$
and
\[ u^* = g^* f(x_1^* + p(\omega)x_2^*), \quad \text{with material constants } C_{ijks}, \quad (2.28) \]

we consider the third formalism,
\[ u^* = g^* f(x_1^* + p(\omega)x_2^*), \quad \text{with material constants } C_{ijks}. \quad (2.29) \]

In the above \( x_1^*, x_2^* \) can be expressed as
\[ x_1^* = \tilde{a} \cdot \tilde{n}, \quad x_2^* = \tilde{a} \cdot \tilde{m}, \quad (2.30) \]

where
\[ \tilde{n}(\omega) = (\cos \omega, \sin \omega, 0), \]
\[ \tilde{m}(\omega) = (-\sin \omega, \cos \omega, 0), \quad (2.31a) \]

or
\[ n_j = \Omega_{1j}, \quad m_j = \Omega_{2j}, \quad (2.31b) \]

and \( \omega \) is an arbitrary real parameter. With (2.29), equation (2.6) is replaced by
\[ Q_{ik}(\omega) = C_{ijks}n_jn_s, \]
\[ R_{ik}(\omega) = C_{ijks}n_jm_s, \quad (2.32) \]
\[ T_{ik}(\omega) = C_{ijks}m_jm_s, \]

and (2.15)-(2.17) become
\[ N(\omega) \xi = p(\omega) \xi, \quad (2.33) \]
\[ N(\omega) = \begin{bmatrix} N_1(\omega) & N_2(\omega) \\ N_3(\omega) & N_1^T(\omega) \end{bmatrix}, \quad \xi = \begin{bmatrix} a \\ \tilde{b} \end{bmatrix}, \quad (2.34) \]
\[ N_1(\omega) = -\tilde{T}^{-1}(\omega)R^T(\omega), \]
\[ N_2(\omega) = \tilde{T}^{-1}(\omega) = \tilde{T}^T(\omega), \quad (2.35) \]
\[ N_3(\omega) = \tilde{R}(\omega)\tilde{T}^{-1}(\omega)R^T(\omega) - Q(\omega) = \tilde{N}^T(\omega) \]

10
It can be shown that $\bar{\zeta}$ is independent of $\omega$ when $\bar{N}(\omega)$ is simple or semi-simple [37]. From (2.21)-(2.23), (2.32)-(2.34) and (2.31b), one can also prove that

$$\bar{Q}(\omega) = \Omega^T(\omega)\bar{Q}^*(\omega)\Omega(\omega),$$

$$\bar{R}(\omega) = \Omega^T(\omega)\bar{R}^*(\omega)\Omega(\omega),$$

$$\bar{T}(\omega) = \Omega^T(\omega)\bar{T}^*(\omega)\Omega(\omega),$$

(2.36)

$$\bar{N}_i(\omega) = \Omega^T(\omega)\bar{N}_i^*(\omega)\Omega(\omega), \quad i = 1, 2, 3.$$  \hspace{1cm} (2.37)

It should be pointed out that (2.32)-(2.35) are reduced to (2.6), (2.15)-(2.17) when $\omega = 0$. Therefore, one can write $\bar{N}_i(0), p(0)$ as $\bar{N}_i, p$. Moreover, from (2.37) we see that $\bar{N}_i(0)$ can also be written as $\bar{N}_i$. By comparison of the two different formalisms in (2.27) and (2.29), and with the fact that displacement $\bar{u}$ is the same in both formalisms, we obtain

$$p(\omega) = \frac{p(0) \cos \omega - \sin \omega}{p(0) \sin \omega + \cos \omega}.$$  \hspace{1cm} (2.38)

With $A$ and $B$ defined in (2.11b), and using the orthogonality relation [36] we obtain three real matrices $\bar{H}, \bar{L}, \bar{S}$.

$$\bar{H} = 2i\bar{A}\bar{A}^T,$$

$$\bar{L} = -2i\bar{B}\bar{B}^T,$$

(2.39)

$$\bar{S} = i(2\bar{A}\bar{B}^T - \bar{I}),$$

where $\bar{I}$ is the unit matrix. It can be shown that $\bar{H}$ and $\bar{L}$ are symmetric and positive definite [37]. Referring to a properly chosen basis, it is shown in [38] that $\bar{H}, \bar{L}$ and $\bar{S}$ have simple structures. The explicit expressions for $\bar{S}, \bar{H}$ and $\bar{L}$ of the orthotropic materials or monoclinic materials with the symmetry plane at $x_3 = 0$ have been obtained by Dongye and Ting [39], and Ting [40]. These three real matrices are not entirely independent. They
are related by the following identities [33]

\[ L\tilde{S} + \tilde{S}^T L = 0, \]

\[ S\tilde{L}^{-1} + \tilde{L}^{-1} S^T = 0, \]

\[ H\tilde{S}^T + \tilde{S}H = 0, \quad (2.40a) \]

\[ S^T \tilde{H}^{-1} + \tilde{H}^{-1} S = 0, \]

\[ H\tilde{L} - S\tilde{S} = I. \]

The first, third and the fifth of (2.40a) can be combined and written as

\[
\begin{bmatrix}
S & H \\
-\tilde{L} & \tilde{S}^T
\end{bmatrix}
\begin{bmatrix}
* & \tilde{S} \\
-\tilde{L} & \tilde{S}^T
\end{bmatrix}
= 
\begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}.
\]

(2.40b)

Barnett and Lothe [41] proposed an integral formalism for determining \( S, \tilde{S}, \tilde{H} \) and \( L \) without finding the eigenvectors \( a_\alpha \) and \( b_\alpha \). They showed that

\[ S = \frac{1}{\pi} \int_{\omega_\alpha}^{\omega_\alpha + \pi} N_1(\omega) d\omega, \]

\[ \tilde{H} = \frac{1}{\pi} \int_{\omega_\alpha}^{\omega_\alpha + \pi} N_2(\omega) d\omega, \quad (2.41) \]

\[ L = -\frac{1}{\pi} \int_{\omega_\alpha}^{\omega_\alpha + \pi} N_3(\omega) d\omega, \]

where \( \omega_\alpha \) is an arbitrary real constant because \( N_i(\omega) \) are periodic in \( \omega \) with periodicity \( \pi \).
CHAPTER 3
ANALYTICAL SOLUTIONS FOR INFINITE BODIES

3.1 Interface Crack Problems

Consider an arbitrary number of collinear cracks lying along the interface of two dissimilar anisotropic materials. The materials are assumed to be perfectly bonded at all points of the interface \( x_2 = 0 \) except those lying in the region of cracks \( L \) (see Figure 1), which are defined by the intervals

\[
a_j \leq x_1 \leq b_j, \quad j = 1, 2, ..., n,
\]

with

\[
-\infty < a_1 < b_1 < a_2 < b_2 < ... < a_n < b_n < \infty.
\]

On the upper and lower surfaces of the cracks, an arbitrary and self-equilibrated loading is specified. In the following, all quantities such as displacements, stress functions, ..., etc., pertaining to the \( x_2 > 0 \) and \( x_2 < 0 \) will be marked with subscripts 1 and 2, respectively.

From Chapter 2, we know that the solution to an individual problem in two-dimensional anisotropic elasticity can be reduced to finding the complex function vector \( \mathbf{f}_j \), which should satisfy the boundary conditions of that problem. In the case of two different materials, however, the elastic properties are discontinuous across the bonded line, and a complete solution to the problem requires the knowledge of two complex function vectors \( \mathbf{f}_1 \) and \( \mathbf{f}_2 \). In order to use the method of analytical continuation, the general solutions for the interface crack problems are chosen to be in the form of (2.11a) and expressed as

\[
\begin{align*}
y_1 &= A_1 f_1(z) + \overline{A_1 f_1(z)}, \\
\phi_1 &= B_1 f_1(z) + \overline{B_1 f_1(z)},
\end{align*}
\]

\[ z \in S_1 \quad (3.1a) \]
and

\[
\begin{align*}
\phi_2 &= B_2 f_2(z) + \overline{B_2 f_2(z)}, \\
\phi_2' &= \overline{\phi_2'}, \\
\end{align*}
\]

where \( f_1(z) \) and \( f_2(z) \) are holomorphic in the regions of \( S_1 \) and \( S_2 \), respectively. The two complex function vectors \( f_1 \) and \( f_2 \) are sought to satisfy continuity of displacement and traction across the bonded portion of the interface, as well as the prescribed-traction condition on the crack portion, i.e.,

\[
\begin{align*}
\begin{cases}
\phi_1 = \phi_2, & x_1 \not\in L, \\
\phi_1' = \phi_2' = \overline{\phi_2'}, & x_1 \in L,
\end{cases}
\end{align*}
\]

where prime (') denotes differentiation with respect to its argument. The last equality comes from the relation (2.14), and \( \overline{\phi}_2 \) is the prescribed traction applied on the upper and lower surfaces of the cracks. When the points along the crack surfaces are considered, integration of the last equation of (3.2) provides \( \phi_1 = \phi_2 \) where the integration constants are neglected since the constant stress function does not produce stresses. Combining this result with the continuity requirement along the bonded portion, we have

\[
\phi_1 = \phi_2, \quad \text{along the entire interface.}
\]

By applying the general solutions given in (3.1), the traction continuity condition leads to

\[
B_1 f_1(x_1^+) - \overline{B_2 f_2(x_1^-)} = B_2 f_2(x_1^-) - \overline{B_1 f_1(x_1^+)},
\]

where \( x_1^+, x_1^- \) denote, respectively, the points on the upper and lower surfaces of the cracks.

One of the important properties of holomorphic functions used in the method of analytical continuation is that if \( f(z) \) is holomorphic in \( S_1 \) (or \( S_2 \)), then \( \overline{f(\overline{z})} \) is holomorphic in \( S_2 \).
(or $S_1$). From this property and equation (3.3), we may introduce a function which is holomorphic in the entire domain including the interface, i.e.,

$$
\tilde{\theta}(z) = \begin{cases} 
B_1 \tilde{f}_1(z) - B_2 \tilde{f}_2(z), & z \in S_1, \\
B_2 \tilde{f}_2(z) - B_1 \tilde{f}_1(z), & z \in S_2.
\end{cases} 
$$

(3.4)

Since $\tilde{\theta}(z)$ is now holomorphic and single-valued in the whole plane including the point at infinity, by Liouville’s Theorem we have $\tilde{\theta}(z) \equiv \text{constant}$. However, constant function vector $\tilde{f}$ corresponds to rigid body motion which may be neglected. Therefore,

$$
\tilde{\theta}(z) \equiv 0. 
$$

(3.5)

Combining (3.4) and (3.5), we have

$$
\tilde{f}_2(z) = B_2^{-1} B_1 \tilde{f}_1(z), \quad z \in S_1,
$$

$$
\tilde{f}_1(z) = B_1^{-1} B_2 \tilde{f}_2(z), \quad z \in S_2. 
$$

(3.6)

For the displacement continuity requirement, substitution of (3.1) and (3.6) into (3.2) gives

$$
\tilde{B}_1 \tilde{f}_1(x_1^+) = \tilde{M}^* \tilde{M}_1^{-1} \tilde{M}_2^{-1} \tilde{B}_2 \tilde{f}_2(x_1^-), \quad x_1 \notin L 
$$

(3.7)

where

$$
\tilde{M}^* = \tilde{M}_1^{-1} + \tilde{M}_2^{-1} 
$$

(3.8)

and $\tilde{M}_k$, $k = 1, 2$, is the impedance matrix [42] defined as

$$
\tilde{M}_k = -iB_k \tilde{A}_k^{-1}, \quad k = 1, 2. 
$$

(3.9)

It has been shown that [36]

$$
\tilde{M}_k = H_k^{-1}(I + iS_k) = (I - iS_k^T)H_k^{-1}, 
$$

$$
\tilde{M}_k^{-1} = L_k^{-1}(I + iS_k^T) = (I - iS_k)L_k^{-1}, 
$$

(3.10)
where $\mathcal{S}_k, \mathcal{H}_k, \mathcal{L}_k$ are three real matrices defined in (2.39) or (2.41). By using (3.10), $\mathcal{M}^*$ defined (3.8) can also be written as

$$
\mathcal{M}^* = -i(\mathcal{W} + i\mathcal{D}) = i(A_1B_1^{-1} - \bar{A}_2\bar{B}_2^{-1}),
$$

(3.11a)

where

$$
\mathcal{W} = \mathcal{S}_1\mathcal{L}_1^{-1} - \mathcal{S}_2\mathcal{L}_2^{-1},
\mathcal{D} = \mathcal{L}_1^{-1} + \mathcal{L}_2^{-1}.
$$

(3.11b)

Similar to the displacement continuity requirement, the prescribed traction condition now provides

$$
B_1\mathcal{f}_1(x_1^+) + B_2\mathcal{f}_2(x_1^-) = \mathcal{f}, \quad x_1 \in L.
$$

(3.12)

If we introduce a new function vector $\mathbf{\psi}(z)$,

$$
\mathbf{\psi}(z) = \begin{cases} 
B_1\mathcal{f}_1(z), & z \in S_1, \\
\mathcal{M}^*^{-1}\mathcal{M}^*B_2\mathcal{f}_2(z), & z \in S_2,
\end{cases}
$$

(3.13)

equations (3.7) and (3.12) lead to the following Hilbert problem [30]

$$
\mathbf{\psi}(x_1^+) = \mathbf{\psi}(x_1^-), \quad x_1 \notin L,
$$

$$
\mathbf{\psi}'(x_1^+) + \mathcal{M}^*^{-1}\mathcal{M}^*\mathbf{\psi}'(x_1^-) = \mathcal{f}(x_1), \quad x_1 \in L.
$$

(3.14)

The solution to this Hilbert problem is (see Appendix)

$$
\mathbf{\psi}'(z) = \frac{1}{2\pi i}X_o(z)\int_{L} \frac{1}{s-z} [X_o^+(s)]^{-1} \mathcal{f}(s)ds + X_o(z)\mathcal{P}_n(z)
$$

(3.15a)

where $\mathcal{P}_n(z)$ is an arbitrary polynomial vector with the degree not higher than $n$, and $X_o(z)$ is the basic Plemelj function satisfying

$$
X_o^+(x_1) = X_o^-(x_1), \quad x_1 \notin L,
$$

$$
X_o^+(x_1) + \mathcal{M}^*^{-1}\mathcal{M}^*X_o^-(x_1) = 0, \quad x_1 \in L,
$$

(3.15b)
i.e.,
\[ X_{\circ}(z) = \Lambda \Gamma(z), \]
\[ \Lambda = [\lambda_1 \quad \lambda_2 \quad \lambda_3], \]
\[ \Gamma(z) = \left\langle \prod_{j=1}^{n} (z - a_j)^{-1+\delta_{\alpha}} (z - b_j)^{\delta_{\alpha}} \right\rangle. \]  

The angular bracket \( \left\langle \right\rangle \) stands for the diagonal matrix, i.e.,
\[ \left\langle f_{\alpha} \right\rangle = \text{diag}[f_1, f_2, f_3] \]

which will be used throughout this report. \( \delta_{\alpha} \) and \( \lambda_\alpha \), \( \alpha = 1, 2, 3 \), of (3.15c) are the eigenvalues and eigenvectors of
\[ (\bar{M}^* + e^{2i\pi \delta} \bar{M}^*) \lambda = 0. \]  

The explicit solution for the eigenvalue \( \delta \) has been given by Ting [33] as
\[ \delta_{\alpha} = -\frac{1}{2} + i \epsilon_{\alpha}, \quad \alpha = 1, 2, 3, \]  

where
\[ \epsilon_1 = \epsilon = \frac{1}{2\pi} \ln \frac{1+\beta}{1-\beta}, \quad \epsilon_2 = -\epsilon, \quad \epsilon_3 = 0, \]
\[ \beta = \left[ -\frac{1}{2} \text{tr}(\bar{W} \bar{D}^{-1}) \right]^{\frac{1}{2}}, \]

\( \text{tr} \) stands for the trace of matrix.

Once we get the solution of \( \psi'(z) \) from (3.15), the complex function \( f_1(z) \) and \( f_2(z) \) can be obtained from (3.13) with the understanding that the subscripts of \( z \) are dropped, and a replacement of \( z_1, z_2 \) or \( z_3 \) should be made for each component function. This calculation procedure will be applied throughout this report. The whole field solution can then be found by using equation (3.1). If one is interested in the stresses \( \sigma_{12} \) and the crack opening displacements \( \Delta \gamma \), the following results show that they have a simple relation.

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with function $\tilde{\psi}(z)$. By applying (2.9)$_2$, (3.1a)$_2$, (3.6)$_2$ and (3.13), the stresses $\sigma_{i2}$ are calculated as

$$
\begin{pmatrix}
\sigma_{12} \\
\sigma_{22} \\
\sigma_{32}
\end{pmatrix} = \beta \frac{f_1}{f_2} \tilde{f}_2^\prime (z) = \beta \frac{f_1}{f_2} \tilde{f}_2^\prime (z) = (\tilde{f}_1 + \tilde{M}^{-1} \tilde{M}^*) \tilde{\psi}^\prime (z).
$$

(3.16)

From (3.1), (3.6) and (3.13), the crack opening displacements $\Delta \tilde{u}$ can also be calculated and simplified as

$$
\Delta \tilde{u} = \tilde{u}(x_1, 0^+) - \tilde{u}(x_1, 0^-),
$$

$$
= -i \tilde{M}^* [\tilde{\psi}(x_1^+) - \tilde{\psi}(x_1^-)], \quad x_1 \in L.
$$

(3.17)

3.2 Fracture Parameters

The application of fracture mechanics bears largely upon the stress intensity factors $K$, crack opening displacements $\Delta \tilde{u}$ and energy release rate $G$. To calculate these parameters, we concentrate on a crack tip region which is small compared with the whole body but sufficiently large with respect to atomic dimensions for us to be reasonably happy with the application of linear elasticity theory. The physical problem considered is therefore a semi-infinite traction-free interface crack. For this problem, the basic Plemelj function given in (3.15b) is

$$
X_\circ (z) = \tilde{\circ} \ll z^{-\frac{1}{2} + i\epsilon_0} \gg.
$$

The complex function $\tilde{\psi}^\prime (z)$ obtained in (3.15a) for $\tilde{\circ} = 0$ is

$$
\tilde{\psi}^\prime (z) = X_\circ (z) \tilde{p}_\circ = \tilde{\circ} \ll z^{-\frac{1}{2} + i\epsilon_0} \gg \tilde{p}_\circ.
$$

(3.18a)

By integration, we have

$$
\tilde{\psi} (z) = \tilde{\circ} \ll \frac{2z^{\frac{1}{2} + i\epsilon_0}}{1 + 2i\epsilon_0} \gg \tilde{p}_\circ.
$$

(3.18b)
By using (3.16), (3.18a) and \( z = r e^{i0} \), the stresses \( \sigma_{ij} \) (or \( \vec{\sigma}' \)) ahead of the crack tip are obtained as

\[
\begin{pmatrix}
\sigma_{12} \\
\sigma_{22} \\
\sigma_{32}
\end{pmatrix}
= \vec{\sigma}' = r^{-\frac{1}{2}}(I + \overline{M}^{-1} \overline{M}^* )\overline{\lambda} \ll r^{i\epsilon_\alpha} \gg \rho_0.
\]  

(3.19a)

Similarly, from (3.17), (3.18b) and \( z = r e^{\pm i\pi} \) for the upper and lower surfaces, we have

\[
\Delta \gamma = 4r^\frac{1}{2} \overline{M}^* \overline{\lambda} \ll \frac{(\cosh \pi \epsilon_\alpha)r^{i\epsilon_\alpha}}{1 + 2i\epsilon_\alpha} \gg \rho_0.
\]  

(3.19b)

The eigenvectors \( \lambda_\alpha \) are fully determined by the eigenvalue problem given in (3.15d) up to an arbitrary complex constant. From (3.18) and (3.19), we see that the constant can be absorbed into the coefficient \( \rho_0 \), which is determined by the external geometry and load. Hence, the normalization of eigenvectors will not affect the final results. Knowing that \( \overline{A}^T \overline{D} \overline{A} \) (or \( \frac{1}{2} \overline{A}^T (\overline{M}^* + \overline{M}^*) \overline{A} \)) is a diagonal matrix [29], normalization of \( \overline{\lambda} \) can be performed by defining

\[
\frac{1}{2} \overline{A}^T (\overline{M}^* + \overline{M}^*) \overline{A} = I, \text{ or } \overline{A}^T \overline{D} \overline{A} = \overline{I}.
\]  

(3.20)

Using the explicit solution (3.15e) for \( \delta_\alpha \), the eigenvalue problem of (3.15d) can also be written as

\[
\overline{M}^* \overline{A} = \overline{M}^* \overline{\lambda} \ll e^{2i\pi \epsilon_\alpha} \gg .
\]  

(3.21)

By applying the normalization defined in (3.20) and using (3.21), the following equalities, which are useful for the simplification of equations, can be obtained as

\[
\begin{align*}
\overline{A}^T \overline{M}^* \overline{A} &= \ll 1 - \tanh(\pi \epsilon_\alpha) \gg , \\
\overline{A}^T \overline{M}^* \overline{A} &= \ll 1 + \tanh(\pi \epsilon_\alpha) \gg , \\
\frac{1}{2} \overline{A}^T (\overline{M}^* - \overline{M}^*) \overline{A} &= -i \overline{A}^T \overline{W} \overline{A} = \ll -\tanh(\pi \epsilon_\alpha) \gg , \\
(\overline{I} + \overline{M}^{-1} \overline{M}^*) \overline{A} &= 2 \overline{A} \ll e^{-\pi \epsilon_\alpha} \cosh(\pi \epsilon_\alpha) \gg .
\end{align*}
\]  

(3.22)
Note that \( \tanh(\pi \varepsilon) = \beta \) by (3.15f). Substituting (3.22) into (3.19), we have

\[
\phi' = 2r^{-\frac{1}{2}} \Lambda \ll e^{-\pi \varepsilon a} (\cosh \pi \varepsilon a)^{r \varepsilon a} \gg \Psi_o,
\]

\[
\Delta \gamma = 4r^{\frac{1}{2}} \Lambda^{-T} \ll \frac{e^{-\pi \varepsilon a \pi \varepsilon a}}{1 + 2i \varepsilon a} \gg \Psi_o.
\]

By applying the virtual crack closure method [43], the total strain energy release rate \( G \) can be calculated as

\[
G = \lim_{\Delta a \to 0} \frac{1}{2\Delta a} \int_0^{\Delta a} \Delta \gamma^T(s - \Delta a) \phi'(s) ds
\]

\[
= 2\pi \Psi_o^T \ll e^{-2\pi \varepsilon a} \gg \Psi_o
\]

where \( s \) is the distance ahead of the crack tip. In the derivation of (3.24), the fact that the displacement is real (hence \( \Delta \gamma = \overline{\Delta \gamma} \)) has been used and the integration is performed by knowing that

\[
\int_0^{\Delta a} \left( \frac{\Delta a - s}{s} \right)^{\frac{1}{2} - i\varepsilon} ds = \frac{\pi \Delta a}{\cosh(\pi \varepsilon a)} \left( \frac{1}{2} - i\varepsilon \right).
\]

To have a proper definition of the stress intensity factor, we recast the solution (3.23) into the form which looks like the classical near tip solution. Introducing

\[
\kappa = 2\sqrt{2\pi} \ll e^{-\pi \varepsilon a} \cosh(\pi \varepsilon a) \gg \Psi_o,
\]

the near tip solutions \( \phi' \), \( \Delta \gamma \) and the energy release rate \( G \) can now be written as

\[
\phi' = \frac{1}{\sqrt{2\pi r}} \Lambda \ll r \varepsilon a \gg \kappa,
\]

\[
\Delta \gamma = \sqrt{\frac{2r}{\pi}} \Lambda^{-T} \ll \frac{r \varepsilon a}{(1 + 2i \varepsilon a) \cosh(\pi \varepsilon a)} \gg \kappa,
\]

\[
G = \frac{1}{4} \kappa^{-T} \ll \frac{1}{\cosh^2(\pi \varepsilon a)} \gg \kappa,
\]

which is the same as Suo [44]. From the eigenvalue problem (3.15d, e) and the normalization defined in (3.20), one can show that \( \lambda_2 = \overline{\lambda_1} \) and \( \lambda_3 \) is real. By (3.26), it can also be
proved that \( k_2 = \overline{k}_1 \) and \( k_3 \) is a real number because the traction \( \phi' \) is real. Hence, the components of \( \sim \) can be expressed as

\[
\sim = \left\{ \begin{array}{c}
 k \\
 \overline{k} \\
 k_{III}
\end{array} \right\}, \quad k = k_I + i k_{II}.
\] (3.27)

An alternative expression for \( \phi' \) can now be written as [44]

\[
\phi' = \frac{1}{\sqrt{2\pi r}} (k r^{ie} \xi_1 + \overline{k} r^{-ie} \overline{\xi}_1 + k_{III} \lambda_3),
\]

from which \( \frac{k r^{ie}}{\sqrt{2\pi r}}, \frac{\overline{k} r^{-ie}}{\sqrt{2\pi r}}, \frac{k_{III}}{\sqrt{2\pi r}} \) can be thought of as the components of the traction \( \phi' \) in the directions of \( \lambda_1, \overline{\lambda}_1 \) and \( \lambda_3 \), respectively. Evaluations of these components can be taken by the following matrix products, i.e.,

\[
\frac{k r^{ie}}{\sqrt{2\pi r}} = \lambda_1^T D \phi', \quad \frac{k_{III}}{\sqrt{2\pi r}} = \lambda_3^T D \phi'.
\]

Therefore, the coefficients \( \sim \) can be explained as the intensity of singularity of the stresses \( \sigma_{i2} \) in the direction of \( \lambda_1, \overline{\lambda}_2 \) and \( \lambda_3 \) and are defined as

\[
\sim = \lim_{r \to 0} \frac{1}{\sqrt{2\pi r}} \ll r^{-ie} \gg \sim^T D \phi',
\] (3.28a)

or

\[
\sim = \lim_{r \to 0} \frac{1}{\sqrt{2\pi r}} \ll r^{-ie} \gg \lambda_3^{-1} \phi'.
\] (3.28b)

Since \( k_I, k_{II} \) and \( k_{III} \) are the scalars measuring the intensity of singularity of the stresses \( \sigma_{i2} \), they may be treated as the stress intensity factors. However, as a consequence of the peculiar singularity, \( \sim \) has an awkward physical unit. A remedy suggested by Rice [27] is to appeal to a fixed length \( ell \) and use the combination \( \ll \ell^{ie} \gg \sim \) as the basic parameter which has the dimension of conventional stress intensity factors, i.e.,

\[
\sim = \ll \ell^{ie} \gg \sim = \lim_{r \to 0} \frac{1}{\sqrt{2\pi r}} \ll (r/\ell)^{-ie} \gg \lambda_3^{-1} \phi'.
\] (3.28c)
The stress intensity factors so defined cannot be reduced to classical stress intensity factors for a crack tip in a homogeneous anisotropic medium when the two materials become the same, because the directions of $\lambda_1, \lambda_2$ are usually not the same as the direction of the crack. To have a comparable definition, we may transform $\sim \kappa$ into the direction of $x_1, x_2$ by

$$\sim \kappa = \Lambda \sim \kappa.$$  \hspace{1cm} (3.29a)

Hence,

$$\sim K = \lim_{r \to 0} \sqrt{2\pi r} \sim \Lambda \ll (r/\ell)^{-i\epsilon_\alpha} \gg \Lambda^{-1} \phi',$$  \hspace{1cm} (3.29b)

where

$$\sim \kappa = \begin{cases} \sim K_{II} \\ \sim K_I \\ \sim K_{III} \end{cases}.$$  \hspace{1cm} (3.29c)

Similar to $R(c)$ defined by Wu [29], it can be shown that $\Lambda \ll (r/\ell)^{-i\epsilon_\alpha} \gg \Lambda^{-1}$ is a real matrix and can be calculated by

$$\Lambda \ll c_\alpha \gg \Lambda^{-1} = I + \frac{1}{\beta^2} \frac{c_R}{c_I} (D^{-1} W)^2 - \frac{c_I}{\beta} D^{-1} W,$$  \hspace{1cm} (3.30)

where $c_1 = c, c_2 = \bar{c}, c_3 = \bar{c}$ and $c_R, c_I$ are real and imaginary parts of $c$. Therefore, $K_I, K_{II}, K_{III}$ are real scalars and may also be treated as the stress intensity factors which will be reduced to the classical stress intensity factors for a homogeneous medium. The near tip solutions and energy release rate in terms of $\sim \kappa$ can then be expressed as

$$\phi' = \frac{1}{\sqrt{2\pi r}} \Lambda \ll (r/\ell)^{i\epsilon_\alpha} \gg \Lambda^{-1} \sim \kappa,$$

$$\Delta \sim u = \sqrt{\frac{2r}{\pi}} \Lambda^{-T} \ll \frac{(r/\ell)^{i\epsilon_\alpha}}{(1 + 2i\epsilon_\alpha) \cosh(\pi \epsilon_\alpha)} \gg \Lambda^{-1} \sim \kappa,$$  \hspace{1cm} (3.31)

$$G = \frac{1}{4} \sim K^{-T} \sim \Lambda^{-T} \ll \frac{1}{\cosh^2(\pi \epsilon_\alpha)} \gg \Lambda^{-1} \sim \kappa.$$
By (3.20) and (3.30), the relation between energy release rate $G$ and stress intensity factors $\widetilde{K}$ shown in (3.31) can be simplified as

$$G = \frac{1}{4} \widetilde{K}^T \tilde{E} \widetilde{K}, \quad \widetilde{E} = \widetilde{D} + \widetilde{W} \widetilde{D}^{-1} \widetilde{W},$$

(3.32)

which is equivalent to the one given by Wu [29].

Although the explicit form of the total strain energy release rate $G$ has been obtained in the previous calculation, the Mode I ($G_1$) and Mode II ($G_2$) strain energy release rates for an interfacial crack may not exist due to the oscillatory characteristics near the crack tip [45, 46]. To get a meaningful $G_1$ and $G_2$, one may consider a partially closed interface crack [7], or by

$$G_1 = \frac{1}{2\Delta a} \int_0^{\Delta a} \Delta u_2(s - \Delta a) \sigma_{22}(s) ds$$

$$G_2 = \frac{1}{2\Delta a} \int_0^{\Delta a} \Delta u_1(s - \Delta a) \sigma_{12}(s) ds$$

$$G_3 = \frac{1}{2\Delta a} \int_0^{\Delta a} \Delta u_3(s - \Delta a) \sigma_{32}(s) ds$$

(3.33)

where $\Delta a$ is a small but finite characteristic length which may be material-dependent. By a similar calculation as $G$, we obtain

$$G_i = 2\pi \frac{\vec{P}_x}{\vec{P}_o} P_{ij}^* P_{ij},$$

$$= \frac{1}{4} \tilde{K}^T \tilde{J}_i \tilde{K},$$

$$= \frac{1}{4} \tilde{K}^T \tilde{E}_i \tilde{K}, \quad i = 1, 2, 3,$$

(3.34a)

where

$$\tilde{J}_i = \frac{2}{\pi \Delta a} \int_0^{\Delta a} \sqrt{\frac{\Delta a - s}{s}} \approx \frac{e^{-\pi \varepsilon \alpha (\Delta a - s)^{-i \varepsilon \alpha}}}{1 - 2i \varepsilon \alpha} \approx \Lambda^{-1} \tilde{J}_i \Lambda \approx e^{-\pi \varepsilon \alpha (\cosh \pi \varepsilon \alpha) s^{i \varepsilon \alpha}} ds,$$

$$\tilde{J}_i = \frac{2}{\pi \Delta a} \int_0^{\Delta a} \sqrt{\frac{\Delta a - s}{s}} \approx \frac{(\varepsilon \alpha - i \varepsilon \alpha)}{(1 - 2i \varepsilon \alpha) \cosh \pi \varepsilon \alpha} \approx \Lambda^{-1} \tilde{J}_i \Lambda \approx s^{i \varepsilon \alpha} ds,$$

$$\tilde{E}_i = \Lambda^{-T} \tilde{J}_i \Lambda^{-1},$$

(3.34b)
and

\[
\mathcal{I}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{I}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{I}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.34c}
\]

### 3.3 Explicit Solutions for Orthotropic Bimaterials

The results presented in Section 3.2 show that the influence of the material properties on the fracture parameters is transmitted through the eigenvalues \( \delta_\alpha \) (hence \( \epsilon_\alpha \)) and eigenvector matrix \( \mathcal{A} \). The explicit solution for the eigenvalues \( \delta_\alpha \) has been given in (3.15e,f) by Ting [33], which is related to the matrices \( \mathcal{W} \) and \( \mathcal{D} \), or \( \mathcal{S} \) and \( \mathcal{L} \) by (3.11b). The explicit expression for \( \mathcal{S}, \mathcal{H} \) and \( \mathcal{L} \) of orthotropic materials has also been shown by Dongye and Ting [39] as

\[
S_{21} = \left[ \frac{C_{66} (\sqrt{C_{11} C_{22}} - C_{12})}{C_{22} (C_{12} + 2C_{66} + \sqrt{C_{11} C_{22}})} \right]^{\frac{1}{2}}, \quad S_{12} = -\sqrt{\frac{C_{22}}{C_{11}}} S_{21},
\]

\[
H_{11} = \sqrt{\frac{C_{22}}{C_{11}}} H_{22}, \quad H_{22} = \left[ \frac{\sqrt{C_{11} C_{22} + C_{66}}}{C_{66} (\sqrt{C_{11} C_{22}} - C_{12})} \right] S_{21}, \quad H_{33} = L_{33}^{-1}, \tag{3.35}
\]

\[
L_{11} = \left( C_{12} + \sqrt{C_{11} C_{22}} \right) S_{21}, \quad L_{22} = \sqrt{\frac{C_{22}}{C_{11}}} L_{11}, \quad L_{33} = (C_{44} C_{55})^{\frac{1}{2}},
\]

and all the other components of \( S_{ij}, H_{ij} \) and \( L_{ij} \) are equal to zero. In the above \( C_{ij} \) is the contracted notation for the 4th order elasticity tensor \( C_{ijkl} \) [47]. In order to have an explicit form solution in terms of the engineering constants, the following expressions for the nonzero stiffness matrix \( C_{ij} \) [48],

\[
C_{11} = \frac{1 - \nu_{23} \nu_{32}}{E_2 E_3 \Delta}, \quad C_{22} = \frac{1 - \nu_{13} \nu_{31}}{E_1 E_3 \Delta}, \quad C_{33} = \frac{1 - \nu_{12} \nu_{21}}{E_1 E_2 \Delta},
\]

\[
C_{12} = C_{21} = \frac{\nu_{21} + \nu_{31} \nu_{23}}{E_2 E_3 \Delta} = \frac{\nu_{12} + \nu_{32} \nu_{13}}{E_1 E_3 \Delta},
\]

\[
C_{13} = C_{31} = \frac{\nu_{31} + \nu_{21} \nu_{32}}{E_2 E_3 \Delta} = \frac{\nu_{13} + \nu_{12} \nu_{23}}{E_1 E_2 \Delta}, \tag{3.36a}
\]

\[
C_{23} = C_{32} = \frac{\nu_{32} + \nu_{12} \nu_{31}}{E_1 E_3 \Delta} = \frac{\nu_{23} + \nu_{21} \nu_{13}}{E_1 E_2 \Delta},
\]

\[
C_{44} = G_{23}, \quad C_{55} = G_{31}, \quad C_{66} = G_{12},
\]

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and

$$\Delta = (1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13})/E_1E_2E_3,$$  \hspace{1cm} (3.36b)

are substituted into (3.35) for the generalized plane strain condition ($\varepsilon_3 = \partial u_3/\partial x_3 = 0$) which is the requirement implied in (2.4). In (3.36), $E_1, E_2, E_3$ are the Young’s moduli in 1, 2 and 3 directions, respectively; $\nu_{ij}$ are the Poisson’s ratios for the transverse strain in the $j-$direction when stressed in the $i-$direction; $G_{23}, G_{31}, G_{12}$ are the shear moduli in the 2-3, 3-1, and 1-2 plane, respectively.

To apply the Stroh’s formalism to the generalized plane stress problems ($\sigma_3 = 0$), the following substitution should be made

$$\hat{C}_{ij} = C_{ij} - C_{i3}C_{j3}/C_{33},$$ \hspace{1cm} (3.37a)

which gives

$$\hat{C}_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad \hat{C}_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}},$$

$$\hat{C}_{12} = \hat{C}_{21} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}},$$ \hspace{1cm} (3.37b)

$$\hat{C}_{44} = G_{23}, \quad \hat{C}_{55} = G_{31}, \quad \hat{C}_{66} = G_{12},$$

Equation (3.37a) comes from the plane stress assumption $\sigma_3 = 0$, i.e., $C_{13}\varepsilon_1 + C_{23}\varepsilon_2 + C_{33}\varepsilon_3 = 0$ for orthotropic materials. Replacement of $\varepsilon_3$ by $\varepsilon_1$ and $\varepsilon_2$ makes the reduced constitutive laws (without $\varepsilon_3$) be related by $\hat{C}_{ij}$ not by $C_{ij}$. In Stroh’s formalism, the displacements are considered to be dependent upon $x_1$ and $x_2$ only, i.e., $\varepsilon_3$ is neglected. Therefore, by the reduced constitutive laws, we use $C_{ij}$ in the generalized plane strain problems and $\hat{C}_{ij}$ in the generalized plane stress problems. Note that $\hat{C}_{ij}$ can also be determined by the inversion of the reduced compliance, i.e.,

$$\hat{C}_{ij}^{-1} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}.$$ \hspace{1cm} (3.37c)

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Since $\sigma_3 = 0$ for the generalized plane stress problems, it may conclude that $\varepsilon_3 \neq 0$ which violates the assumption of Stroh's formalism. To have a valid application of the Stroh's formalism, the displacements and stresses have been explained as the average values through the thickness of the plate, which depend only on $x_1$ and $x_2$.

Substituting (3.36) or (3.37) into (3.35), we obtain

$$ S_{12} = -\alpha_2 \kappa_1 \nu_1, \quad S_{21} = \alpha_1 \kappa_2 \nu_1, $$

$$ H_{11} = \alpha_2 \kappa_1 \nu_2 / G_{12}, \quad H_{22} = \alpha_1 \kappa_2 \nu_2 / G_{12}, \quad H_{33} = 1 / \sqrt{G_{23} G_{31}}, \quad (3.38a) $$

$$ L_{11} = \alpha_1 \kappa_1 E_1, \quad L_{22} = \alpha_2 \kappa_2 E_2, \quad L_{33} = \sqrt{G_{23} G_{31}}, $$

where

$$ \kappa_1 = (E_1 / G_{12} + 2 \nu_1 \sqrt{E_1 / E_2})^{-\frac{1}{2}}, \quad (3.38b) $$

$$ \kappa_2 = (E_2 / G_{12} + 2 \nu_1 \sqrt{E_2 / E_1})^{-\frac{1}{2}}, $$

and

$$ \alpha_1 = (1 - \nu_{13} \nu_{31})^{-\frac{1}{2}} \quad \alpha_2 = (1 - \nu_{23} \nu_{32})^{-\frac{1}{2}} $$

$$ \nu_1 = \sqrt{(1 - \nu_{13} \nu_{31})(1 - \nu_{23} \nu_{32}) - \sqrt{(\nu_{21} + \nu_{31} \nu_{23})(\nu_{12} + \nu_{32} \nu_{13})}}, $$

$$ \nu_2 = \sqrt{(1 - \nu_{13} \nu_{31})(1 - \nu_{23} \nu_{32}) + (1 - \nu_{12} \nu_{21} - \nu_{23} \nu_{32} - \nu_{31} \nu_{13} - 2 \nu_{21} \nu_{32} \nu_{13}) G_{12} / \sqrt{E_1 E_2}} \quad (3.38c) $$

for plane strain conditions, and

$$ \alpha_1 = 1, \quad \alpha_2 = 1, $$

$$ \nu_1 = 1 - \sqrt{\nu_{12} \nu_{21}}, $$

$$ \nu_2 = 1 + (1 - \nu_{12} \nu_{21}) G_{12} / \sqrt{E_1 E_2} \quad (3.38d) $$

for plane stress conditions. For isotropic materials, $E_1 = E_2 = E, G_{12} = G_{23} = G_{31} = \mu = \frac{E}{2(1+\nu)}, \nu_{23} = \nu_{32} = \nu_{13} = \nu_{31} = \nu_{12} = \nu_{21} = \nu$, these three fundamental elasticity matrices
can be simplified as,
\[
S_{21} = -S_{12} = \frac{1 - 2\nu}{2(1 - \nu)},
\]
\[
H_{11} = H_{22} = \frac{3 - 4\nu}{4\mu(1 - \nu)}, \quad H_{33} = \frac{1}{\mu}, \quad (3.39a)
\]
\[
L_{11} = L_{22} = \frac{E}{2(1 - \nu^2)} = \frac{\mu}{1 - \nu}, \quad L_{33} = \frac{E}{2(1 + \nu)} = \mu,
\]
for plane strain conditions, and
\[
S_{21} = -S_{12} = \frac{1 - \nu}{2},
\]
\[
H_{11} = H_{22} = \frac{3 - \nu}{4\mu}, \quad H_{33} = \frac{1}{\mu}, \quad (3.39b)
\]
\[
L_{11} = L_{22} = \frac{E}{2} = \mu(1 + \nu), \quad L_{33} = \mu,
\]
for plane stress conditions. With the explicit forms of $S, H$ and $L$ given in (3.38), the matrices $\tilde{W}$ and $\tilde{D}$ can also be expressed in terms of the engineering constants, by (3.11b) we have
\[
D_{11} = (\alpha_1 \kappa_1 E_1)^{-1} + (\alpha_1 \kappa_1 E_1)^{-1},
\]
\[
D_{22} = (\alpha_2 \kappa_2 E_2)^{-1} + (\alpha_2 \kappa_2 E_2)^{-1},
\]
\[
D_{33} = (G_{23} G_{31})^{-\frac{1}{2}} + (G_{23} G_{31})^{-\frac{1}{2}}, \quad (3.40a)
\]
\[
W_{21} = -W_{12} = (\kappa_2 \nu_1 / \kappa_1 E_1)^{1} - (\kappa_2 \nu_1 / \kappa_1 E_1)^{2},
\]
and all the other components of $D_{ij}$ and $W_{ij}$ are equal to zero. The subscripts (1) and (2) denote, the properties of materials 1 and 2, respectively. For isotropic materials,
\[
D_{11} = D_{22} = \frac{1 - \nu_1}{\mu_1} + \frac{1 - \nu_2}{\mu_2}, \quad D_{33} = \frac{1}{\mu_1} + \frac{1}{\mu_2}, \quad (3.40b)
\]
\[
W_{21} = -W_{12} = \frac{1 - 2\nu_1}{2\mu_1} - \frac{1 - 2\nu_2}{2\mu_2},
\]
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for plane strain conditions, and
\[ D_{11} = D_{22} = \frac{1}{\mu_1(1 + \nu_1)} + \frac{1}{\mu_2(1 + \nu_2)}, \quad D_{33} = \frac{1}{\mu_1} + \frac{1}{\mu_2}, \]
\[ W_{21} = -W_{12} = \frac{1 - \nu_1}{2\mu_1(1 + \nu_1)} - \frac{1 - \nu_2}{2\mu_2(1 + \nu_2)}. \]  
(3.40c)

for plane stress conditions. The explicit form of bimaterial matrix \( \mathcal{M}^* \) for orthotropic bimaterials defined in (3.8) or (3.11) can now be written as
\[
\mathcal{M}^* = \begin{bmatrix}
D_{11} & iW_{21} & 0 \\
-iW_{21} & D_{22} & 0 \\
0 & 0 & D_{33}
\end{bmatrix}.
\]  
(3.41a)

The eigenvalue \( \epsilon \) given in (3.15f) is then simplified as
\[
\epsilon = \frac{1}{2\pi} \ln \frac{1 + \beta}{1 - \beta}, \quad \beta = |W_{21}(D_{11}D_{22})^{-\frac{1}{2}}|,
\]  
(3.41b)

where the positive value of \( \beta \) is chosen. By applying (3.41), the eigenvector matrix \( \mathcal{A} \) can be obtained explicitly from the eigenvalue problem given in (3.15), as
\[
\mathcal{A} = \begin{bmatrix}
-i\sqrt{2D_{11}} & i\sqrt{2D_{11}} & 0 \\
D_{11}(D_{11}d_1) & D_{11}(D_{11}d_2) & 0 \\
0 & 0 & d_3
\end{bmatrix},
\]  
(3.42a)

where \( d_1, d_2, d_3 \) are arbitrary constants which will be determined by the normalization defined in (3.20). After normalization, we have
\[
\mathcal{A} = \begin{bmatrix}
-i\sqrt{2D_{11}} & i\sqrt{2D_{11}} & 0 \\
1/\sqrt{2D_{22}} & 1/\sqrt{2D_{22}} & 0 \\
0 & 0 & 1/\sqrt{D_{33}}
\end{bmatrix},
\]  
(3.42b)

and
\[
\mathcal{A} \ll c_{\alpha} \mathcal{A}^{-1} = \begin{bmatrix}
c_R & c_1 \sqrt{D_{22}/D_{11}} & 0 \\
-c_1 \sqrt{D_{11}/D_{22}} & c_R & 0 \\
0 & 0 & 1
\end{bmatrix},
\]  
(3.43)

which can also be derived by (3.30). With \( \mathcal{A} \) given in (3.42b) for orthotropic bimaterials, \( J_1^*, J_2 \) and \( E_2 \) of (3.34b) can be simplified as
\[
\tilde{J}_1 = \frac{1}{2} \begin{bmatrix}
e^{-2\pi\epsilon} & \overline{\epsilon} & 0 \\
\overline{\epsilon} & e^{2\pi\epsilon} & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \tilde{J}_2 = \frac{1}{2} \begin{bmatrix}
e^{-2\pi\epsilon} & -\overline{\epsilon} & 0 \\
-\overline{\epsilon} & e^{2\pi\epsilon} & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \tilde{J}_3 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]
(3.44)
\[
J_1 = \frac{1}{2 \cosh^2 \pi \epsilon} \begin{bmatrix} 1 & \Upsilon & 0 \\ \overline{\Upsilon} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad J_2 = \frac{1}{2 \cosh^2 \pi \epsilon} \begin{bmatrix} 1 & -\Upsilon & 0 \\ -\overline{\Upsilon} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
E_1 = \frac{1}{2 \cosh^2 \pi \epsilon} \begin{bmatrix} D_{11} - \frac{1}{2}(\Upsilon + \overline{\Upsilon}) & -i \sqrt{D_{11}D_{22}}(\Upsilon - \overline{\Upsilon}) & 0 \\ -i \sqrt{D_{11}D_{22}}(\Upsilon - \overline{\Upsilon}) & D_{22} + \frac{1}{2}(\Upsilon + \overline{\Upsilon}) & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
E_2 = \frac{1}{2 \cosh^2 \pi \epsilon} \begin{bmatrix} D_{11} + \frac{1}{2}(\Upsilon + \overline{\Upsilon}) & i \sqrt{D_{11}D_{22}}(\Upsilon - \overline{\Upsilon}) & 0 \\ i \sqrt{D_{11}D_{22}}(\Upsilon - \overline{\Upsilon}) & D_{22} - \frac{1}{2}(\Upsilon + \overline{\Upsilon}) & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
E_3 = D_{33} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{(3.44a)}
\]

where
\[
\Upsilon = \frac{2 \cosh(\pi \epsilon)}{\pi \Delta a (1 - 2i \epsilon)} \int_0^{\Delta a} (\Delta a - s)^{\frac{1}{2} - i \epsilon} s^{\frac{1}{2} - i \epsilon} ds. \quad \text{(3.44b)}
\]

Note that \( \Upsilon \) may not exist for \( \Delta a \rightarrow 0 \) due to the oscillatory characteristics near the crack tip. By (3.32), (3.34a), and (3.44) the explicit expressions for \( G_i, i=1,2,3 \), and \( G \) in terms of the stress intensity factors \( K_I, K_{II} \) and \( K_{III} \) can be written as

\[
G_1 = \frac{1}{8 \cosh^2 \pi \epsilon} \{ D_{22}[1 - \frac{1}{2}(\Upsilon + \overline{\Upsilon})]K_I^2 + D_{11}[1 - \frac{1}{2}(\Upsilon + \overline{\Upsilon})]K_{II}^2 - i \sqrt{D_{11}D_{22}}(\Upsilon - \overline{\Upsilon})K_I K_{II} \},
\]

\[
G_2 = \frac{1}{8 \cosh^2 \pi \epsilon} \{ D_{22}[1 - \frac{1}{2}(\Upsilon + \overline{\Upsilon})]K_I^2 + D_{11}[1 + \frac{1}{2}(\Upsilon + \overline{\Upsilon})]K_{II}^2 + i \sqrt{D_{11}D_{22}}(\Upsilon - \overline{\Upsilon})K_I K_{II} \},
\]

\[
G_3 = \frac{1}{4}D_{33}K_{III}^2,
\]

\[
G = G_1 + G_2 + G_3 = \frac{1}{4 \cosh^2 \pi \epsilon} \{ D_{22}K_I^2 + D_{11}K_{II}^2 \} + \frac{1}{4}D_{33}K_{III}^2. \quad \text{(3.45a)}
\]

For homogeneous orthotropic plates, we have \( W_{21} = 0, \epsilon = 0, \Upsilon = 1 \) by (3.40), (3.41b) and (3.44b). The energy release rates are reduced to

\[
G_1 = \frac{D_{22}}{4}K_I^2, \quad G_2 = \frac{D_{11}}{4}K_{II}^2, \quad G_3 = \frac{D_{33}}{4}K_{III}^2, \quad \text{(3.45b)}
\]

\[
G = \frac{1}{4}(D_{22}K_I^2 + D_{11}K_{II}^2 + D_{33}K_{III}^2).
\]
3.4 Composite Laminates

Consider a laminated composite composed of unidirectional fiber-reinforced plies. The composite laminate is considered to be sufficient long so that, in the region far away from the ends, end effects are negligible by virtue of Saint Venant principle. Consequently, stresses in the laminate are independent of the $x_3$-axis. The case in which stresses and displacements are independent of $x_3$ is the generalized plane deformation problem considered in this Chapter. For the purpose of illustrating the fundamental behavior of delamination fracture in composite laminates, only two plies with different fiber orientations containing delamination cracks along the $\theta^+$ and $\theta^-$ ply interface are studied (Figure 2). Lamina properties typical of high-modulus unidirectional composite for aircraft construction are assumed to be

$$E_f = E_1, \quad E_t = E_h = E_2,$$

$$G_{ft} = G_{fh} = G_{th} = \mu,$$

$$\nu_{ft} = \nu_{fh} = \nu_{th} = \nu,$$ \hspace{1cm} (3.46a)

where $E$, $\mu$ and $\nu$ are the Young’s modulus, shear modulus and Poisson’s ratio, respectively. The subscripts $f$, $t$ and $h$ refer to the fiber, transverse and through-thickness directions, respectively. By the symmetry restrictions on the elastic constants, we have

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}$$

and hence,

$$\nu_{tf} = \nu_{hf} = \nu \frac{E_2}{E_1}, \quad \nu_{th} = \nu.$$ \hspace{1cm} (3.46b)

Substituting (3.46) into (3.36), the nonzero stiffnesses in the principal material directions
$(f, t, h)$ can be simplified as

$$
C'_{11} = \frac{1 - \nu^2}{E'_{2} \Delta}, \quad C'_{22} = C'_{33} = \frac{E'_{1} - \nu^2 E_{2}}{E'_{2} \Delta},
$$

$$
C'_{12} = C'_{13} = \frac{\nu(1 + \nu)}{E_{1} E_{2} \Delta}, \quad C'_{23} = \frac{\nu(E_{1} + \nu E_{2})}{E'_{1} E_{2} \Delta}, \tag{3.47a}
$$

$$
C'_{44} = C'_{55} = C'_{66} = \mu,
$$

where

$$
\Delta = \frac{(1 + \nu)[(1 - \nu)E_{1} - 2\nu^2 E_{2}]}{(E_{1} E_{2})^2}. \tag{3.47b}
$$

If the principal material directions $(f, t, h)$ of orthotropy do not coincide with coordinate directions $(x_1, x_2, x_3)$ that are geometrically nature to the solution of the problem, a transformed stiffness $C_{ij}$ in the coordinate directions is needed. Because $C_{ijkl}$ represents a 4th order tensor, the transformation law is

$$
C'_{pqrs} = \alpha_{ip} \alpha_{jq} \alpha_{kr} \alpha_{ls} C_{ijkl}, \quad \alpha_{mn} = \cos(x'_m, x_n). \tag{3.48}
$$

Consider an orthotropic lamina with fiber lying on the $x_1 x_3$ plane oriented counterclockwise an angle $\theta$ from $x_1$-axis. By using the transformation law (3.48) and the relation between tensor and contracted notation, we obtain the stiffness $C_{ij}$ in the coordinate directions as,

$$
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0 \\
C_{12} & C_{22} & C_{23} & 0 & C_{25} & 0 \\
C_{13} & C_{23} & C_{33} & 0 & C_{35} & 0 \\
0 & 0 & 0 & C_{44} & 0 & C_{46} \\
C_{15} & C_{25} & C_{35} & 0 & C_{55} & 0 \\
0 & 0 & 0 & C_{46} & 0 & C_{66}
\end{bmatrix}, \tag{3.49a}
$$

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where

\[ C_{11} = C'_{11} \cos^4 \theta + 2(C'_{13} + 2C'_{55}) \cos^2 \theta \sin^2 \theta + C'_{33} \sin^4 \theta, \]

\[ C_{12} = C'_{12} \cos^2 \theta + C'_{23} \sin^2 \theta, \]

\[ C_{13} = (C'_{11} + C'_{33} - 4C'_{55}) \cos^2 \theta \sin^2 \theta + C'_{13}(\cos^4 \theta + \sin^4 \theta), \]

\[ C_{15} = (C'_{11} - C'_{13} - 2C'_{55}) \cos^3 \theta \sin \theta + (C'_{13} - C'_{33} + 2C'_{55}) \cos \theta \sin^3 \theta, \]

\[ C_{22} = C'_{22}, \]

\[ C_{23} = C'_{12} \sin^2 \theta + C'_{23} \cos^2 \theta, \]

\[ C_{25} = (C'_{12} - C'_{23}) \cos \theta \sin \theta, \]

\[ C_{33} = C'_{11} \sin^4 \theta + 2(C'_{13} + 2C'_{55}) \cos^2 \theta \sin^2 \theta + C'_{33} \cos^4 \theta, \]

\[ C_{35} = (C'_{11} - C'_{13} - 2C'_{55}) \cos \theta \sin^3 \theta + (C'_{13} - C'_{33} + 2C'_{55}) \cos^3 \theta \sin \theta, \]

\[ C_{44} = C'_{44} \cos^2 \theta + C'_{66} \sin^2 \theta, \]

\[ C_{46} = (C'_{66} - C'_{44}) \cos \theta \sin \theta, \]

\[ C_{55} = (C'_{11} + C'_{33} - 2C'_{13} + 2C'_{55}) \cos^2 \theta \sin^2 \theta + C'_{55}(\cos^4 \theta + \sin^4 \theta), \]

\[ C_{66} = C'_{66} \cos^2 \theta + C'_{44} \sin^2 \theta. \]

Because the stiffness \( C_{ij} \) in (3.49) is a material property of monoclinic medium with symmetry plane at \( z_2 = 0 \), the explicit expressions (3.35) for the specially orthotropic materials cannot be used. To calculate the fundamental elasticity matrices \( S, H \) and \( L \), the definition given in (2.15) can be applied, in which \( \tilde{A} \) and \( \tilde{B} \) are computed from (2.11b), (2.5) and (2.8). However, this definition is valid only for the material eigenvalues \( \rho_\alpha \) are distinct or the eigenvectors \( \tilde{a}_\alpha \) (or \( \tilde{b}_\alpha \)) are independent each other. For the cases that \( \tilde{a}_\alpha \) (or \( \tilde{b}_\alpha \))
are not independent, a modified definition has been given in [49] as

\[ S = i(2A^Y B^T - I), \]

\[ H = 2iA^Y A^{T}, \]

\[ L = -2i
\]

where

\[ Y = \begin{bmatrix} 0 & 1 & 0 \\ 1 & \delta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \delta = p_2 - p_1, \]

\[ A' = [a_1^o \quad a_2^o \quad a_3], \quad B' = [b_1^o \quad b_2^o \quad b_3], \]

and \((a_1^o, b_1^o), (a_2^o, b_2^o), (a_3, b_3)\) are determined by

\[ N_{\tilde{\xi}^1} = p_1 \tilde{\xi}^1, \]

\[ N_{\tilde{\xi}^2} = p_2 \tilde{\xi}^2 + \tilde{\xi}^1, \]

\[ N_{\tilde{\xi}^3} = p_3 \tilde{\xi}^3, \]

\[ \tilde{\xi}_1^o = \left\{ \begin{array}{c} a_1^o \\ b_1^o \end{array} \right\}, \quad \tilde{\xi}_2^o = \left\{ \begin{array}{c} a_2^o \\ b_2^o \end{array} \right\}, \quad \tilde{\xi}_3 = \left\{ \begin{array}{c} a_3 \\ b_3 \end{array} \right\}. \]

The eigenvectors are obtained by the eigenvalue problem (3.50d) up to an arbitrary complex constant which is determined by the following orthonormalization

\[ A'^T B' + B'^T A' = Y^{-1}, \]

\[ A'^T \tilde{Y}' + \tilde{Y}' A' = 0. \]

Instead of finding the 6-vector \( \xi \) from (3.50d) one could find the 3-vectors \( a \) and \( b \) by the way similar to (2.5) and (2.8). This may have some advantages in a numerical calculation because not only the matrix \( D(p) \) is smaller than \( \tilde{N} \), one dose not have to find the inverse \( T^{-1} \) as shown in (2.17). When \( \tilde{N} \) is non-semisimple, so is \( D(p) \) of equation (2.5). This
means that when $p_2 = p_1$, $q_2 = q_1$. For the cases when $D(p)$ is non-semisimple or almost non-semisimple, equations (2.5) and (2.8) are modified as [49]

\[
\begin{align*}
D(p_1) a_i^o &= 0, \quad (3.52a) \\
D(p_2) a_i^o + [(R + R^T) + (p_1 + p_2)T] a_i^o &= 0,
\end{align*}
\]

and

\[
\begin{align*}
\tilde{b}_1^o &= (\tilde{R}^T + p_1 \tilde{T}) a_i^o = -\left(\frac{1}{p_1} \tilde{Q} + \tilde{R}\right) a_i^o, \\
\tilde{b}_2^o &= (\tilde{R}^T + p_2 \tilde{T}) a_i^o + \tilde{T} a_i^o = -\left(\frac{1}{p_2} \tilde{Q} + \tilde{R}\right) a_i^o + \frac{1}{p_1 p_2} \tilde{Q} a_i^o. \quad (3.52b)
\end{align*}
\]

An alternative approach to find $S$, $H$ and $L$, which can also avoid the degeneracy problem is the integral formalism introduced by Barnett and Lothe [41] as shown in (2.41). The advantage of this integral formalism is that the calculation of eigenvalues and eigenvectors is unnecessary. However, a numerical inversion and integral should be performed, which is time consuming.

When the fundamental matrices $S$, $H$ and $L$ are calculated numerically based upon the stiffness $C_{ij}$ given in (3.49), (3.47) and (3.46), the matrices $W$, $D$ and $M^*$ can be obtained by (3.11). The eigenvalues $\delta_\alpha$ (or $\epsilon_\alpha$) and eigenvectors $\chi_\alpha$ are then found by (3.15d,e,f). With these solutions, the near tip solutions and energy release rate can be calculated by (3.31) and (3.34).

### 3.5 Illustrations

The solution obtained in (3.15) is valid for an arbitrary number of collinear interface cracks and an arbitrary loading condition. In the following, three important examples are presented for illustration. The first is a finite crack subjected to uniform loading (Figure 3), the second is a semi-infinite crack subjected to a point load (Figure 4), and the last is a finite crack subjected to a point load (Figure 5).
(i) A Finite Crack Subjected to Uniform Loading

Consider an interface crack located on \( a_1 = -a, b_1 = a \), subjected to uniform loading \( \mathbf{\tilde{\gamma}} = \{\tilde{\sigma}_{12}, \tilde{\sigma}_{22}, \tilde{\sigma}_{23}\}^T (= \text{constant}) \). To find the solution for this problem, the line integral given (3.15a) should be evaluated first. By residue theory, the integral around a closed contour \( C \) shown in Figure 3 can be calculated as

\[
\oint \frac{1}{s-z} [X_0(s)]^{-1} \mathbf{\tilde{\gamma}} ds = 2\pi i \sum_{k=1}^{n} \mathbf{r}_k,
\]

\[
X_0(s) = \frac{1}{\sqrt{s^2 - a^2}} \Lambda \ll (\frac{s-a}{s+a})^{i\epsilon_0} \gg,
\]

where \( \mathbf{r}_k \) are the residues of the integrand at its singular points within \( C \). The closed contour \( C \) is the union of \( L^+, C_0, L^-, L_1, C_\infty, L'_1, C'_0 \). The summation of the integrals along \( L_1 \) and \( L'_1 \) vanishes since they have opposite directions and the integrand across this line is continuous. The integrals around the circle \( C_0 \) and \( C'_0 \) can be proved to be zero when the radii of the circles \( C_0 \) and \( C'_0 \) tend to zero. By replacing the contour of \( C_\infty \) as \( Re^{i\psi} \) and letting \( R \to \infty \), the integral around \( C_\infty \) is obtained to be

\[
\int_{C_\infty} \frac{1}{s-z} [X_0(s)]^{-1} \mathbf{\tilde{\gamma}} ds = 2\pi i \ll z + 2i\epsilon_0 a \gg \Lambda^{-1} \mathbf{\tilde{\gamma}}.
\]  

(3.54a)

If we let

\[
Y_\sim(z) = \int_{L^+} \frac{1}{s-z} [X_0^+(s)]^{-1} ds,
\]

(3.55a)

through the use of (3.15b) \( 2 \), we have

\[
\int_{L'^+ + L'^-} \frac{1}{s-z} [X_0(s)]^{-1} \mathbf{\tilde{\gamma}} ds = Y_\sim(z)(I - \mathbf{M}^{-1} \mathbf{M}^*) \mathbf{\tilde{\gamma}}.
\]

(3.55b)

The only pole which has contribution to the residues is at \( s = z \), and the residue at that point is \([X_0(z)]^{-1} \mathbf{\tilde{\gamma}} \). We are now in a position to evaluate the line integral and the final simplified result is

\[
\psi'(z) = \Lambda \ll 1 - \frac{z + 2i\epsilon_0 a}{\sqrt{z^2 - a^2}} (\frac{z-a}{z+a})^{i\epsilon_0} \gg \Lambda^{-1} (I + \mathbf{M}^{-1} \mathbf{M}^*)^{-1} \mathbf{\tilde{\gamma}}.
\]

(3.56)

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With this solution, one can calculate the complex function vectors \( \mathbf{f}_1 \) and \( \mathbf{f}_2 \) by (3.13), and the displacements and stresses are determined by (3.1). The stress intensity factors \( \tilde{K} \) defined in (3.29) can then be obtained by applying (3.16), (3.22), (3.56) and considering \( z = r + a, \ r \to 0 \), which are

\[
\tilde{K} = -\sqrt{\pi a}\Lambda \ll (1 + 2i\epsilon_o)(2a/\ell)^{-i\epsilon_o} \gg \Lambda^{-1}\tilde{\mathbf{t}}. \tag{3.57}
\]

For orthotropic bimaterials, by use of (3.43), the explicit solution of \( \tilde{K} \) can be derived as

\[
K_I = -\sqrt{\pi a}\{\sigma_{22}[\cos(\epsilon n2a/\ell) + 2\epsilon \sin(\epsilon n2a/\ell)] + \sigma_{12}\sqrt{D_{11}/D_{22}}[\sin(\epsilon n2a/\ell) - 2\epsilon \cos(\epsilon n2a/\ell)]\},
\]

\[
K_{II} = -\sqrt{\pi a}\{\sigma_{12}[\cos(\epsilon n2a/\ell) + 2\epsilon \sin(\epsilon n2a/\ell)] - \sigma_{22}\sqrt{D_{22}/D_{11}}[\sin(\epsilon n2a/\ell) - 2\epsilon \cos(\epsilon n2a/\ell)]\},
\]

\[
K_{III} = -\sqrt{\pi a}\sigma_{32}. \tag{3.58}
\]

The stress intensity factors for isotropic bimaterials are obtained by letting \( D_{11} = D_{22} \), which are equivalent to those defined by Rice and Sih [22] except a constant factor \( \cosh \pi \epsilon \).

The results can also be reduced to the classical stress intensity factors for a crack tip in a homogeneous anisotropic medium in which \( \epsilon = 0 \) by (3.15f) with \( W = 0 \).

(ii) A Semi-infinite Crack Subjected to a Point Load

Let the semi-infinite planes of different materials be joined along the positive \( x_1 \)-axis.

A line crack is situated along the negative \( x_1 \)-axis extending from \( x_1 = 0 \) to \( x_1 = -\infty \) and is opened by a point force \( \mathbf{t}_o \) at \( x = -a \) on each side of the crack (Figure 4). For this problem, the Plemelj function \( X_o(z) \) used is

\[
X_o(z) = \Lambda \ll z^{-\frac{1}{2} + i\epsilon_o} \gg. \tag{3.59}
\]

The point load \( \mathbf{t}(s) \) can be represented by a delta function, i.e.,

\[
\hat{\mathbf{t}}(s) = \delta(s + a)\mathbf{t}_o, \quad \mathbf{t}_o = \begin{pmatrix} \tilde{t}_1 \\ \tilde{t}_2 \\ \tilde{t}_3 \end{pmatrix}. \tag{3.60}
\]
Substituting (3.59) and (3.60) into (3.15a), we have

$$
\psi'(z) = -\frac{1}{2\pi i(z + a)} X_o(z)[X_o^+(-a)]^{-1} t_o, \quad (3.61a)
$$

or

$$
\psi'(z) = \frac{1}{2\pi (z + a)} \ll e^{\pi i \alpha} \left( \frac{a}{z} \right)^{\frac{1}{2} - i \alpha} \gg \Lambda^{-1} t_o. \quad (3.61b)
$$

By a similar approach as (i), we obtain the stress intensity factors $K$ as

$$
\tilde{K} = \sqrt{\frac{2}{\pi a}} \ll \frac{a}{\ell} \cos \pi \alpha \gg \Lambda^{-1} t_o. \quad (3.62)
$$

For orthotropic bimaterials, we have

$$
K_I = \sqrt{\frac{2}{\pi a}} \cos \pi \epsilon \left[ \hat{t}_2 \cos(\ell \ln a/\ell) + \hat{t}_1 \sqrt{D_{11}/D_{22}} \sin(\ell \ln a/\ell) \right],
$$

$$
K_{II} = \sqrt{\frac{2}{\pi a}} \cos \pi \epsilon \left[ \hat{t}_1 \cos(\ell \ln a/\ell) - \hat{t}_2 \sqrt{D_{22}/D_{11}} \sin(\ell \ln a/\ell) \right], \quad (3.63)
$$

$$
K_{III} = \sqrt{\frac{2}{\pi a}} \hat{t}_3.
$$

(iii) A Finite Crack Subjected to a Point Load

For this problem, $X_o(z)$ is the same as (3.53)$_2$ while $\tilde{t}$ is represented by

$$
\tilde{t}(s) = \delta(s - c) t_o, \quad (3.64)
$$

where $c$ is the location of the point load (Figure 5). Substituting (3.53)$_2$ and (3.64) into (3.15a), and applying (3.16), (3.22)$_4$ and (3.29), we obtain the stress intensity factors $\tilde{K}$ as

$$
\tilde{K} = -\frac{1}{\sqrt{\pi a}} \sqrt{\frac{a + c}{a - c}} \ll \left[ \frac{\ell(a - c)}{2a(a + c)} \right]^{i \pi \alpha} e^{-2\pi \epsilon \alpha \cosh \pi \epsilon \alpha} \gg \Lambda^{-1} t_o. \quad (3.65)
$$
For orthotropic bimaterials, we have

\[
K_I = -\frac{1}{\sqrt{\pi a}} \sqrt{\frac{a + c}{a - c}} e^{-2\pi \epsilon} \cosh \pi \epsilon \left\{ \hat{t}_2 \cos \left( \frac{2a(a + c)}{\ell(a - c)} \right) + \hat{t}_1 \sqrt{D_{11}/D_{22}} \sin \left( \frac{2a(a + c)}{\ell(a - c)} \right) \right\},
\]

\[
K_{II} = -\frac{1}{\sqrt{\pi a}} \sqrt{\frac{a + c}{a - c}} e^{-2\pi \epsilon} \cosh \pi \epsilon \left\{ \hat{t}_1 \sin \left( \frac{2a(a + c)}{\ell(a - c)} \right) - \hat{t}_2 \sqrt{D_{22}/D_{11}} \sin \left( \frac{2a(a + c)}{\ell(a - c)} \right) \right\},
\]

\[
K_{III} = -\frac{1}{\sqrt{\pi a}} \sqrt{\frac{a + c}{a - c}} \hat{t}_3.
\]

(3.66)
CHAPTER 4

APPROXIMATE SOLUTIONS FOR FINITE BODIES

In this Chapter the general formulation is based upon the following assumptions: (1) the material of each layer can be treated as macroscopically homogeneous, orthotropic and continuous body; (2) the deformation is small such that the infinitesimal elasticity theory of anisotropic body can be applied, and the materials obey generalized Hooke's law; (3) the laminated construction is symmetric; (4) loads are applied in the plane of the structural laminates only, and the distribution of the inplane loads must be symmetric about the central plane; (5) the thickness of the structural laminate is small compared to its other dimensions. Each layer of thickness $2h$ is considered as a three dimension elastic body bounded by the planes $x_2 = \pm h$ and a smooth cylindrical surface $x_1 = x_1(s), x_3 = x_3(s)$. Following the asymptotic analysis introduced by Reiss and Locke [31], the 3D elasticity problems can be treated by two 2D problems, i.e., the interior problems and the boundary layer problems (Figure 6).

4.1 Interior Problems

Since the thickness of the structural laminate is considered to be small compared to its other dimensions, the dimensionless variable $\xi = x_2/h$ is introduced. The equilibrium equations given in (2.3) and the compatibility equations [47] are therefore written as

\[
\begin{align*}
\sigma_{12,\xi} + h(\sigma_{11,1} + \sigma_{13,3}) &= 0, \\
\sigma_{22,\xi} + h(\sigma_{21,1} + \sigma_{23,3}) &= 0, \\
\sigma_{32,\xi} + h(\sigma_{31,1} + \sigma_{33,3}) &= 0,
\end{align*}
\]

(4.1a)
\[ \begin{align*}
\varepsilon_{11,33} + \varepsilon_{33,11} - 2\varepsilon_{13,13} &= 0, \\
\varepsilon_{11,cc} &= 2h\varepsilon_{12,1c} - h^2\varepsilon_{22,11}, \\
\varepsilon_{33,cc} &= 2h\varepsilon_{23,3c} - h^2\varepsilon_{22,33}, \\
\varepsilon_{13,cc} &= h(\varepsilon_{23,1c} + \varepsilon_{13,3c}) - h^2\varepsilon_{22,13}, \\
\varepsilon_{11,3c} - \varepsilon_{13,1c} &= h(\varepsilon_{12,13} - \varepsilon_{23,11}), \\
\varepsilon_{33,1c} - \varepsilon_{13,3c} &= h(\varepsilon_{23,13} - \varepsilon_{12,33}),
\end{align*} \]  

where stresses and strains are functions of \( x_1, \zeta \) and \( x_3 \). If the displacements and stresses are split into the sum of even and odd portions with respect to the variable \( \zeta \) in thickness direction, it has been shown [50] that the equations of the three dimensional linear anisotropic elasticity can be separated into two independent systems. One of the systems has the nine unknowns \( \sigma_{11}^e, \sigma_{22}^e, \sigma_{33}^e, \sigma_{13}^e, u_1^e, u_5^e, \) and \( \sigma_{12}^o, \sigma_{23}^o, u_3^o \) where the superscript \( e \) and \( o \) denote the even and odd portions, respectively. This system constitutes the stretching problem. The remaining components belong to the bending problem. In the following, attention is focused on the stretching problem.

In an asymptotic sense, each stress component can be represented by a power series in \( h \), i.e.,

\[ \sigma_{ij}(x_1, \zeta, x_3; h) = \sum_{k=0}^{\infty} \sigma^{(k)}_{ij}(x_1, \zeta, x_3)h^k. \]  

The functions \( \sigma^{(k)}_{ij} \) are called the interior stress coefficients and are assumed to possess the same even or odd property as \( \sigma_{ij} \). Substituting (4.2) into (4.1a) and neglecting the terms associated with \( h \) since \( h \) is small, we have

\[ \sigma_{12}^{(o)} = \sigma_{12}^{(o)}(x_1, x_3), \quad \sigma_{22}^{(o)} = \sigma_{22}^{(o)}(x_1, x_3), \quad \sigma_{32}^{(o)} = \sigma_{32}^{(o)}(x_1, x_3). \]  

If the top and bottom surfaces of the laminates are assumed to be traction free, equation...
(4.3a) leads to

$$\sigma_{12}^{(o)} = \sigma_{22}^{(o)} = \sigma_{32}^{(o)} = 0.$$  \hfill (4.3b)

Since the stresses and strains are related linearly by the Hooke's law given in (2.2), each strain component can also be represented by a power series in \( h \). If only the stretching problem is concerned, substitution of this power series into (4.1b) leads to

$$\varepsilon_{ij}^{(o)} = \varepsilon_{ij}^{(o)}(x_1, x_2), ij = 11, 33, 13.$$ \hfill (4.4)

From comparison with the plane problem of elasticity, it is noted that the zeroth order solution given in (4.3b) and (4.4) correspond to the classical plane stress problem and can be treated by the classical lamination theory. Because the laminate is presumed to consist of perfectly bonded laminae, the inplane strain components will be the same throughout the thickness direction if only the stretching problem is considered. The stresses in each ply may be calculated from the ply stress-strain law. Since the assumption that \( h \) is small compared to its other dimensions is not valid near the edge, the stress boundary conditions in each ply along the free edge may not be satisfied. They are only satisfied in an integrated manner. For the region away from the edge, i.e., in the 'interior' of the plate, the interior solutions given above provide a good approximation. Near the edge, i.e., in the 'boundary layer', the solutions vary rapidly from those of the interior solution to satisfy the boundary conditions of the exact theory. In order to satisfy the stress boundary conditions exactly, an additional stress field is superimposed, which is discussed next.

### 4.2 Boundary Layer Problems

The additional stress field defined in the boundary layer region must satisfy the stress boundary conditions along the free edge, i.e.,

$$\sigma_{ij} = -\sigma_{ij}^{(o)}, \quad ij = 13, 33,$$ \hfill (4.5)

$$\sigma_{32} = 0,$$

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where $\sigma_{ij}^{(o)}$ are the stresses in each ply along the free edge calculated in the interior problems. Since the interior solutions already provide good approximation for the region far away from the edge, the additional stresses $\sigma_{ij}$ determined from the boundary layer problem should satisfy

$$\lim_{x_3 \to \infty} \sigma_{ij} = 0. \quad (4.6)$$

Here, the $x_3$-axis is taken as the direction normal to the edge, and $x_1$-axis is tangent to the edge. Similar to the analysis of the interior problems, another dimensionless variable $\eta = x_3/h$ is now introduced to imply that the boundary layer effects penetrates from the edge into the plate only a distance of order of the magnitude $h$. The boundary layer stresses may also be expanded in asymptotic power series of $h$ as

$$\sigma_{ij}(x_1, \zeta, \eta; h) = \sum_{k=0}^{\infty} \sigma_{ij}^{(k)}(x_1, \zeta, \eta) h^k, \quad (4.7)$$

where $\sigma_{ij}^{(k)}$ are called the boundary layer stress coefficients. Applying the dimensionless variables $\zeta$ and $\eta$, and substituting (4.7) into the equilibrium equations and compatibility equations given in (4.1) yields the following zeroth order solution;

$$\sigma_{13,\eta}^{(o)} + \sigma_{12,\zeta}^{(o)} = 0,$$

$$\sigma_{23,\eta}^{(o)} + \sigma_{22,\zeta}^{(o)} = 0, \quad (4.8a)$$

$$\sigma_{33,\eta}^{(o)} + \sigma_{32,\zeta}^{(o)} = 0,$$

$$\varepsilon_{22,\eta\eta}^{(o)} + \varepsilon_{33,\zeta\zeta}^{(o)} = 2\varepsilon_{23,\zeta\eta}^{(o)},$$

$$\varepsilon_{13,\zeta\eta}^{(o)} + \varepsilon_{12,\eta\eta}^{(o)} = 0, \quad (4.8b)$$

$$\varepsilon_{13,\zeta\zeta}^{(o)} + \varepsilon_{12,\eta\eta}^{(o)} = 0,$$

$$\varepsilon_{11,\zeta\zeta}^{(o)} = \varepsilon_{11,\eta\eta}^{(o)} = \varepsilon_{11,\zeta\eta}^{(o)} = 0.$$
It may be observed that equilibrium equations (4.8a) are exactly the same as those obtained by Pipes and Pagano [1], in which the stress components are considered to be independent of $x_1$. By (4.8b)$_{2,3}$, we have $\varepsilon_{13,\gamma}^{(o)} - \varepsilon_{12,\eta}^{(o)} = \text{constant}$. If $\varepsilon_{11}^{(o)}$ is an even function of $\zeta$ and vanishes when $\eta \to \infty$, equations (4.8b)$_{4,5,\xi}$ lead to $\varepsilon_{11}^{(o)} = 0$. It can be seen that (4.8a)$_1$ and (4.8b)$_{2,3}$ constitute a torsion problem while (4.8a)$_{2,3}$ and (4.8b)$_1$ can be identified as a plane problem. The general form of the displacement field for the boundary layer problems is therefore identical to that given by Pipes and Pagano [1] as

\[
\begin{align*}
  u_1 &= \varepsilon_0 x_1 + U_1(\zeta, \eta), \\
  u_2 &= U_2(\zeta, \eta), \\
  u_3 &= U_3(\zeta, \eta),
\end{align*}
\]

(4.9)

Note that the uniform axial extension $\varepsilon_0$ is zero due to the result of $\varepsilon_{11}^{(o)} = 0$ given above.

### 4.3 Quasi-3D Finite Element Formulation

In finite element formulation, we consider the displacements at any point inside the element are expressed in terms of the nodal displacements $\mathbf{u}_e$ through a set of shape functions $\mathbf{N}_i$ as

\[
\mathbf{u} = \mathbf{N} \mathbf{u}_e + \mathbf{u}_o.
\]

(4.10a)

If the eight-node isoparametric element is chosen, we have

\[
\mathbf{N} = \begin{bmatrix} N_1 & N_2 & \cdots & N_8 \end{bmatrix}, \quad \mathbf{u}_e = \begin{bmatrix} u_{1e} \\
                  u_{2e} \\
                  \vdots \\
                  u_{8e} \end{bmatrix}, \quad \mathbf{u}_o = \begin{bmatrix} \varepsilon_0 x_1 \\
                                                             0 \\
                                                             0 \end{bmatrix}.
\]

(4.10b)

where

\[
\mathbf{N}_i = N_i \mathbf{J}, \quad \mathbf{u}_i = \begin{bmatrix} u_1 \\
                                                      u_2 \\
                                                      u_3 \end{bmatrix}, \quad i = 1, 2, \ldots, 8,
\]

(4.10c)
and the shape functions $R_i$ of eight-node isoparametric element are

$$
R_1 = \frac{1}{4}(1 + r)(1 + s) - \frac{1}{4}(1 - r^2)(1 + s) - \frac{1}{4}(1 - s^2)(1 + r), \\
R_2 = \frac{1}{4}(1 - r)(1 + s) - \frac{1}{4}(1 - r^2)(1 + s) - \frac{1}{4}(1 - s^2)(1 - r), \\
R_3 = \frac{1}{4}(1 - r)(1 - s) - \frac{1}{4}(1 - r^2)(1 - s) - \frac{1}{4}(1 - s^2)(1 - r), \\
R_4 = \frac{1}{4}(1 + r)(1 - s) - \frac{1}{4}(1 - r^2)(1 - s) - \frac{1}{4}(1 - s^2)(1 + r), \\
R_5 = \frac{1}{2}(1 - r^2)(1 + s), \\
R_6 = \frac{1}{2}(1 - s^2)(1 - r), \\
R_7 = \frac{1}{2}(1 - r^2)(1 - s), \\
R_8 = \frac{1}{2}(1 - s^2)(1 + r),
$$

in which $(r, s)$ are the local curvilinear coordinates ranging from -1 to 1, and the nodal numbers 1 to 8 are arranged in an counterclockwise sequence starting from the corner node of (1,1).

By the strain-displacement relation given in (2.1), the strains within the element can also be expressed in terms of the element nodal displacements as

$$
\varepsilon = \tilde{\mathbf{N}} \tilde{u}_e + \varepsilon_o, 
$$

(4.11a)

where

$$
\tilde{\mathbf{N}} = [\tilde{N}_1 \, \tilde{N}_2 \, \ldots \, \tilde{N}_8], \quad \varepsilon = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{pmatrix}, \quad \varepsilon_o = \begin{pmatrix} \varepsilon_o \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

(4.11b)
and
\[
\hat{N}_i = \begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{\partial N_1}{\partial \xi} & 0 \\
0 & 0 & \frac{\partial N_1}{\partial \eta} \\
0 & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_1}{\partial \xi} \\
\frac{\partial N_2}{\partial \xi} & 0 & 0 \\
\frac{\partial N_2}{\partial \xi} & 0 & 0
\end{pmatrix}.
\]

(4.11c)

The stresses are related to the strains by (2.2) as
\[
\sigma = C \varepsilon = C [\bar{\nabla} u_e + \bar{\varepsilon}_o],
\]

(4.12a)

where
\[
\sigma = \begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{pmatrix},
\]

(4.12b)

and \( C \) has been given in (3.49a) for each lamina. The total potential energy of the elastic body will be the sum of the energy contributions of the individual elements. Thus
\[
\pi = \sum_e \pi_e,
\]

(4.13)

where \( \pi_e \) represents the potential energy of element \( e \) which can be written as
\[
\pi_e = \frac{1}{2} \int_{V_e} \sigma^T \varepsilon \, dV - \int_{S_e} u^T f \, dV - \int_{S_e} u^T t \, dS.
\]

(4.14)

\( \bar{f} \) and \( \bar{t} \) denote the body forces and the surface tractions, respectively. \( V_e \) is the element volume and \( S_e \) is the loaded element surface area. If the body forces are neglected, substituting (4.10)-(4.12) into (4.14) leads to
\[
\pi_e = \frac{1}{2} \int_{V_e} [2 \varepsilon_o^T C \bar{\nabla} u_e + \varepsilon_o^T \bar{\varepsilon}_o + u_e^T \bar{\nabla} C \bar{\varepsilon}_o] \, dV - \int_{S_e} [u_e^T \bar{\nabla}^T \bar{t}_o + u_e^T \bar{t}_o] \, dS.
\]

(4.15)

Performance of the minimization for element \( e \) with respect to the nodal displacement \( \bar{u}_e \) for the element results in
\[
\frac{\partial \pi_e}{\partial \bar{u}_e} = \{ \int_{V_e} \bar{\nabla}^T \bar{C} \bar{\nabla} dV \} \bar{u}_e + \int_{V_e} \bar{\nabla}^T \bar{\varepsilon}_o dV - \int_{S_e} \bar{\nabla}^T \bar{t}_o \, dS = 0,
\]

(4.16a)
or,

\[ K_e u_e = q_e, \quad \text{(4.16b)} \]

where

\[ K_e = \int_{V_e} \tilde{\mathbf{N}}^T \tilde{\mathbf{N}} dV, \quad \text{(4.16c)} \]

\[ q_e = \int_{S_e} \tilde{\mathbf{N}}^T t dS - \int_{V_e} \tilde{\mathbf{N}} \tilde{C} \tilde{\varepsilon}_e dV. \]

\( K_e \) is termed the element stiffness matrix and \( q_e \) is the equivalent nodal force. The summation of the terms in (4.16) over all the elements results in a system of equilibrium equations for the complete continuum. These equations are then solved by any standard technique to yield the nodal displacements. The stresses and strains within each element can then be calculated from the nodal displacements using (4.11) and (4.12).
CHAPTER 5
RESULTS AND DISCUSSIONS

The main purpose of this research is to find out the failure criterion for the delamination. Before starting the experimental work, a search for the possible meaningful fracture parameters is important. By analytical investigation in Chapter 3, two possible definitions for the stress intensity factors are provided in (3.28) and (3.29), which can be explained as the intensity of singularity of the stresses. Due to the oscillatory characteristics near the crack tip, the Mode I and Mode II strain energy release rates for an interfacial crack may not exist [45,46]. To extract valuable information for each failure mode, the definition of $G_i$, $i = 1, 2, 3$, are changed from differential form to incremental form as given in (3.33) and (3.34). To have a better understanding about these definitions, a series of numerical calculations based upon the analytical results are given in the following.

5.1 Stress Intensity Factors

Since the definition of stress intensity factors given in (3.28) cannot be reduced to classical stress intensity factors for a crack in a homogeneous anisotropic medium when the two materials become the same, we consider only the definition given in (3.29). Unlike the homogeneous solids for which the stress intensity factors are in most cases independent of the material constants, the stress intensity factors of interfacial cracks may depend on the material constants of upper and lower media. In other words, the stress intensity factors may be a function of ply orientation in composite laminates. To know the dependency on the ply orientation, the analytical result (3.57) of the stress intensity factor for a finite crack subjected to uniform loading is presented by a series of figures, in which the material
of each lamina is taken as Scotch ply 1002 Glass/Epoxy whose properties are

\[ E_{11} = 38.6 \text{ Gpa}, \quad E_{22} = E_{33} = 8.27 \text{ Gpa}, \]

\[ G_{12} = G_{23} = G_{13} = 4.14 \text{ Gpa}, \]

\[ \nu_{12} = \nu_{23} = \nu_{13} = 0.26. \]  

(5.1)

The detail calculation is based upon the discussions given in Section 3.4. Figures 7-9 show the influence of fiber orientation \((\theta^+/\theta^-)\) upon the stress intensity factors \((K_I, K_{II}, K_{III})\) subjected to inplane shear loading \(\hat{\sigma}_{12}\). \(\theta^+\) and \(\theta^-\) represent, respectively, the fiber direction of upper and lower ply oriented counterclockwise from \(x_1\)-axis. Similarly, for tensile loading \(\hat{\sigma}_{22}\) and antiplane shearing \(\hat{\sigma}_{32}\), the results are shown in Figures 10-15. By a careful investigation and proper curve fitting, we see that Figure 7 can be represented by the following expression

\[ K_I^{(1)}/\hat{\sigma}_{12}\sqrt{\pi a} = [g_1(\theta^+) - g_1(\theta^-)]. \]  

(5.2a)

Similarly, for Figures 11 and 13, we have

\[ K_{II}^{(2)}/\hat{\sigma}_{22}\sqrt{\pi a} = \sqrt{\pi a}[g_2(\theta^+) - g_2(\theta^-)], \]  

\[ K_{II}^{(3)}/\hat{\sigma}_{32}\sqrt{\pi a} = \sqrt{\pi a}[g_3(\theta^+) - g_3(\theta^-)], \]  

(5.2b)

where

\[ g_1(\theta) = (-0.014 + 4.995\theta^2 - 2.027\theta^4 + 0.275\theta^6) \times 10^{-2}, \]

\[ g_2(\theta) = (-0.035 - 7.770\theta^2 + 3.875\theta^4 - 0.657\theta^6) \times 10^{-2}, \]  

(5.2c)

\[ g_3(\theta) = (-2.619\theta + 2.158\theta^3 - 0.452\theta^5) \times 10^{-2}. \]

Although only one variable appears in the expression for function \(g(\theta)\), it also depends on the material properties of each lamina and the reference length \(\ell\) chosen. The above expressions (5.2) are valid only for the material properties given in (5.1) and \(a = \ell\). They
may provide us some useful information when studying the failure criterion of delamination. The approximation of these curve fitted functions has been shown in Figures 16-18. With these approximation, the analytical solutions shown in (3.57) can then be expressed as

\[
\begin{bmatrix}
K_{II} \\
K_I \\
K_{III}
\end{bmatrix} = \sqrt{\pi a} \begin{bmatrix}
1 & g_2(\theta^+) - g_2(\theta^-) & \ast \\
g_1(\theta^+) - g_1(\theta^-) & 1 & g_3(\theta^+) - g_3(\theta^-) \\
\ast & \ast & 1
\end{bmatrix}
\begin{bmatrix}
\hat{\sigma}_{12} \\
\hat{\sigma}_{22} \\
\hat{\sigma}_{32}
\end{bmatrix},
\]

(5.3)

where star (\ast) denotes a certain function which is not shown in this report.

5.2 Energy Release Rates

The value of the incremental form definition of \( G_i \) given in (3.33) depends on the constancy of \( G_i \) with respect to \( \Delta a \). Same as the conclusions given in [45,46], Figures 19-21 show that there exists a range of \( \Delta a/a \) where \( G_1 \) and \( G_2 \) remain relatively constant; that is, if finite crack extensions \( \Delta a \) are chosen from this region for calculation of the strain energy release rates, unambiguous values of \( G_i \) can be obtained. By (3.34) and (3.57), the dependency of \( G_i \) on the ply orientation is then shown in Figures 22-30, in which \( \Delta a \) is chosen to be 0.08\( a \).
CHAPTER 6
CONCLUSIONS

General analytical solutions for the interface cracks between two dissimilar anisotropic media are obtained in this report. Two possible meaningful definitions for the stress intensity factors are provided. A modified energy release rate for each failure mode is defined to avoid any ambiguity. Explicit closed form solutions for the orthotropic bimaterials are given in order to have a better understanding about these newly defined fracture parameters. General procedures for the application of the analytical solutions to composite laminate has also been presented. Numerical studies provide some simple curve-fitted functions, which relate the fracture parameters and ply orientations and may be useful for the future experimental work. By asymptotic analysis, a finite 3D elasticity problem is reduced to two 2D problems, i.e., the interior and boundary layer problems.
REFERENCES


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APPENDIX

SOLUTIONS TO THE HILBERT PROBLEM

Consider the problem,

\[ \psi'(x_1^+) + \overline{M^{*-1}} M^* \psi'(x_1^-) = \overline{\lambda}, \quad x_1 \text{ on } L \text{ except at the ends.} \]  \hspace{1cm} (A.1)

Firstly, a solution will be studied which may have a pole of arbitrary order at infinity, and a beginning will be made with the homogeneous problem,

\[ \psi'(x_1^+) + \overline{M}^{*-1} M^* \psi'(x_1^-) = 0, \quad x_1 \in L. \]  \hspace{1cm} (A.2)

A particular solution \( \psi'_0(z) \) of the problem will be sought in the form

\[ \psi'_0(z) = \prod_{j=1}^{n} (z - a_j)^{-1+\delta}(z - b_j)^{\delta} \lambda, \]  \hspace{1cm} (A.3)

where \( \delta \) is a complex constant and \( \lambda \) is a complex constant vector. The function \( \psi'_0(z) \) is holomorphic in the entire plane cut along \( L \), if a definite branch of this function is selected.

It is readily verified by an investigation of the variation in the argument of \( z - a_j \) or \( z - b_j \), when \( z \) describes a closed path beginning a point \( x_1 \) of the arc \( a_jb_j \) and leading, without intersecting \( L \), from the left side of \( a_jb_j \) around the end \( a_j \) to the right side of the arc or around the end \( b_j \), that

\[ \psi'_0(x_1^+) = e^{2i\pi \delta} \psi'_0(x_1^-). \]  \hspace{1cm} (A.4)

Hence, \( \psi'_0(z) \) will satisfy the boundary condition (A.2), provided

\[ (e^{2i\pi \delta} I + \overline{M}^{*-1} M^*) \lambda = 0, \]  \hspace{1cm} (A.5a)

or

\[ (\overline{M}^* + e^{2i\pi \delta} M^*) \lambda = 0. \]  \hspace{1cm} (A.5b)
The explicit solution for the eigenvalue $\delta$ has been given by Ting [33] as

$$
\delta_1 = -\frac{1}{2} + i\epsilon, \quad \delta_2 = -\frac{1}{2} - i\epsilon, \quad \delta_3 = -\frac{1}{2}, \quad (A.6)
$$

where

$$
\epsilon = \frac{1}{2\pi} \ln \frac{1+\beta}{1-\beta}, \quad \beta = \left[-\frac{1}{2} \text{tr}(WD^{-1})^2\right]^{\frac{1}{2}}, \quad (A.7)
$$

$\text{tr}$ stands for the trace of the matrix. Thus, a particular solution $\psi'_o(z)$ of the homogeneous problem has been found; it is given by (A.3) with $\delta$ and $\lambda$ determined by (A.5). Since there are three eigenvalues to (A.5), a linear combination of these particular solutions will still be one of the particular solutions, i.e.,

$$
\psi'_o(z) = X_o(z) P_o, \quad (A.8a)
$$

where

$$
X_o(z) = \Lambda \Gamma(z), \quad (A.8b)
$$

and

$$
\Lambda = [\lambda_1 \lambda_2 \lambda_3],
$$

$$
\Gamma(z) = \prod_{j=1}^{n} (z - a_j)^{-(1+\delta_a)}(z - b_j)^{\delta_a} \gg . \quad (A.8c)
$$

$P_o$ is a coefficient vector. This particular solution does not vanish anywhere in the finite part of the plane and it is unbounded like $|z - a_j|^{-\frac{1}{2}}$ and $|z - b_j|^{-\frac{1}{2}}$ near the ends $a_j$ and $b_j$ respectively.

The most general solution of the homogeneous problem will now be found which has a pole at infinity. For this purpose it will be noted that $\psi'_o(z) = X_o(z) P_o$, being a solution of the homogeneous problem, satisfies the condition

$$
X_o^+ P_o + \tilde{M}^{-1} M X_o^+ P_o = 0, \quad x_1 \in L
$$

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Hence
\[ \overline{M}^{-1} M = -\overline{X^\dagger_o} [X_o^-]^{-1}, \]  
(A.9)
where $X^\dagger_o$ is the simplified notation for $X_o(x_1^\dagger)$. By applying (A.9), eqn. (A.2) becomes
\[
[X_o^+]^{-1} \psi'(x_1^+) - [X_o^-]^{-1} \psi'(x_1^-) = 0, \quad x_1 \in L, \tag{A.10a}
\]
or
\[
\psi_*(x_1^+) - \psi_*(x_1^-) = 0, \quad x_1 \in L, \tag{A.10b}
\]
where $\psi_*(z)$ denotes the sectionally holomorphic function $[X_o(z)]^{-1} \psi'(z)$. It follows (A.10) that $\psi_*(z)$ is holomorphic in the entire plane, except at the point $z = \infty$, provided it is given suitable values on $L$. Further, since $\psi_*(z)$ can only have a pole at infinity, it must, by the generalized Liouville theorem, be a polynomial. Thus, the most general solution of the homogeneous problem is given by
\[
\psi^{(h)}(z) = X_o(z) P_n(z), \tag{A.11}
\]
where $P_n(z)$ is an arbitrary polynomial vector. If it is desired to obtain a solution which is also holomorphic at infinity, it must be assumed that the degree of the polynomial $P_n(z)$ does not exceed $n$. This follows from the behavior of $X_o(z)$ at infinity as given in (A.8b).

Next consider the non-homogeneous problem. Using (A.9), the boundary condition (A.1) may be written as
\[
[X_o^+]^{-1} \psi'(x_1^+) - [X_o^-]^{-1} \psi'(x_1^-) = [X_o^+]^{-1} \frac{\partial}{\partial x_1}, \quad x_1 \in L, \tag{A.12a}
\]
or
\[
\psi_*(x_1^+) - \psi_*(x_1^-) = [X_o^+]^{-1} \frac{\partial}{\partial x_1}, \quad x_1 \in L, \tag{A.12b}
\]
where $\hat{\psi}^*(x^+_1) = [\hat{X}_o(z)]^{-1}\hat{\psi}'(z)$. Equation (A.12) is the simplest case of Hilbert problem whose solution has been given in [30], i.e.,

$$\hat{\psi}'(z) = \frac{1}{2\pi i} \hat{X}_o(z) \int_L \frac{1}{s - z} [\hat{X}_o^+(s)]^{-1}\hat{\xi}(s)ds + \hat{X}_o(z)\hat{p}_n(z)$$  \hspace{1cm} (4.13)

where $\hat{p}_n(z)$ is an arbitrary polynomial vector with the degree not higher than $n$. 
I:  (Interior Problems)

B:  (Boundary Layer Problems)
\[ G_3 \text{ (Nt-m/m}^2) \]

\[ \theta^- = 0^\circ \]
\[ \theta^- = 15^\circ \]
\[ \theta^- = 30^\circ \]
\[ \theta^- = 45^\circ \]
\[ \theta^- = 60^\circ \]
\[ \theta^- = 75^\circ \]
\[ \theta^- = 90^\circ \]

\[ \theta^+ \text{ (Degree)} \]
$C_{15}^{(N1-m/w^2)}$

\begin{align*}
\theta &= 0^\circ \\
\theta &= 15^\circ \\
\theta &= 30^\circ \\
\theta &= 45^\circ \\
\theta &= 60^\circ \\
\theta &= 75^\circ \\
\theta &= 90^\circ \\
\end{align*}