CORRESPONDENCE RELATIONS BETWEEN ANISOTROPIC AND ISOTROPIC ELASTICITY

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ABSTRACT

A simple correspondence relation between two-dimensional linear anisotropic and isotropic elasticity is provided in this paper through the well known complex variable formulation developed by Muskhelishvili for isotropic media and by Streh for anisotropic media. With this correspondence relation, any newly found anisotropic solutions can easily be checked analytically (without numerical calculation) by the existing corresponding isotropic solutions. Moreover, these relations also suggest an easier way to solve the anisotropic problems by referring to the corresponding isotropic solutions.

Keywords: Anisotropic elasticity, isotropic elasticity, complex variable formulation, plane problems, correspondence relations.

1. INTRODUCTION

One of the frequently encountered problems in elasticity is how to verify the newly found anisotropic solutions by the existing corresponding isotropic solutions. Due to the degeneracy property of isotropic materials, no commonly recognized procedure has been suggested. Only a general reduction principle was provided by Chadwick and Smith [1] and Ting [18] for degenerate materials. The usual way which has now been used depends upon the numerical calculation by introducing a small perturbation in the material eigenvalues, e.g., Hwu and Yen [9], or the analytical derivation of the physical responses along the boundary such as the hoop stress for the hole problems or the stress intensity factors for the crack problems, e.g., Hwu [5]. However, it is dangerous without the whole field comparison. For example, the anisotropic solutions presented by Clements [2] for the thermoelastic interface crack problems, although valid along the interface and crack surface, cannot be applied directly to the full field domain without further notification [6].

In this paper, we try to establish the correspondence relations between anisotropic and isotropic elasticity through three well known complex variable formulations developed by Muskhelishvili [14] for isotropic media, and Lekhnitskii [12,13] and Streh [15,16] for anisotropic media. We hope that through the present simple correspondence relations, any newly found anisotropic solutions can easily be checked analytically (without numerical calculation) by the existing corresponding isotropic solutions. It is also hoped that these relations may suggest an easier way to solve the anisotropic problems by referring to the corresponding isotropic solutions.

In order to demonstrate the usefulness of the present correspondence relations, four typical examples concerning holes, cracks, line forces or dislocations, and punch indentation are illustrated explicitly.

2. COMPLEX VARIABLE FORMULATION

The basic equations of the classical theory of linear elasticity contain the strain-displacement equations, the stress-strain laws, and the equations of equilibrium, which may be expressed as (in a fixed rectangular coordinate system $x_i, i = 1, 2, 3$)

$$
\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad \sigma_{ij} = C_{ijkl} \varepsilon_{kl},
$$

$$
\sigma_{ij} = C_{ijkl} \varepsilon_{kl} + 0, \quad (2.1)
$$

"Professor"

where repeated indices imply summation, a comma stands for differentiation. \( w_j, \sigma_{ij}, e_{ij} \) denotes, respectively, the displacement, stress and strain. \( C_{gij} \) are the elastic constants which are assumed to be fully symmetric and positive definite, and are composed of 21 independent constants for general anisotropic materials. For isotropic materials, only two independent constants are taken in the stress-strain relations which are usually written as

\[
\sigma_{ij} = 2 \mu e_{ij} + \lambda \delta_{ij} e_{kk},
\]

(2.2)

where \( \lambda, \mu \) are the Lame constants, \( \delta_{ij} \) is the Kroenecker delta.

In this paper, we shall be concerned with two-dimensional elastic problems in which \( x_3 \) does not appear in the basic equations or the boundary conditions. These problems are of two distinct types. One is plane problems, the other is antiplane shear problems. Plane problems may further be distinguished by plane strain and generalized plane stress. For isotropic materials, the conversion between plane strain and generalized plane stress is usually made on replacing \( \lambda \) by \( \lambda' = \frac{\lambda}{1-\nu} \) for generalized plane stress, and replacing \( \lambda' \) by \( \lambda \) for plane strain. Generally, for anisotropic materials, the plane strain problems are usually treated by employing \( C_{ij} \) with the 3rd row and 3rd column deleted, or employing \( S_{ij}(=C_{ij}^{-1}) \) where

\[
S_{ij} = S_{ij} - S_{ji}, \quad S_{ij} = \frac{1}{2}(S_{ij} + S_{ji}).
\]

While the plane stress problems are treated by employing \( S_{ij} \) with the 3rd row and 3rd column deleted, or employing \( C_{ij}(=S_{ij}^{-1}) \) where

\[
C_{ij} = C_{ij} - C_{ki} C_{ij} C_{kij}.
\]

Note that the 6x6 matrices \( S_{ij} \) and \( C_{ij} \) are, respectively, the contracted notations of the tensors \( C_{ij3j} \) and \( S_{ij3j} \) \[11]\.

It is shown that the general solution to the above basic equations (2.1) may be expressed in terms of holomorphic functions of complex variables. This enables us to apply many of the powerful results of complex function theory to the problems of two-dimensional elasticity. For the convenience of the following discussion, we now summarize the general solutions developed by Muskhelishvili [14] for isotropic materials, and by Lekhnitskii [12,13] and by Stroh [15,16] for anisotropic materials.

2.1 Muskhelishvili Formulation

For two-dimensional isotropic elastostatic problems, the most powerful and elegant method is the complex variable method developed by Muskhelishvili and his co-workers. In his well known book [4], many solution techniques are developed and many problems have been solved analytically. For plane problems, the general solution provided in the Muskhelishvili formulation may be written as follows.

\[
\sigma_{31} + \sigma_{22} = 4 \text{Re} \left[ \psi'(z) \right],
\]

\[
\sigma_{21} - \sigma_{12} + 2i \tau_{12} = 2 \left[ \psi''(z) + \psi'(z) \right],
\]

\[
2\mu(\mu_{21} + i \mu_{12}) = \kappa \psi(z) - \frac{1}{2} \psi''(z) - \psi'(z),
\]

(2.3)

where \( \kappa \) and \( \psi(z) \) are the holomorphic functions of complex variable \( z = x_1 + x_3 i \), which will be determined by the boundary conditions. \( \kappa = 3 - 4v \) for plane strain and \( \kappa = \frac{3}{2}(1 - v) \) for generalized plane stress where \( v = \frac{x_3}{x_1} \) is Poisson's ratio. Since for most isotropic elastic materials Poisson's ratio takes values in the range \( 0 < \psi < 1/2 \) we see that \( \kappa \) satisfies the inequalities \( 1 < \kappa < 3 \) for both plane strain and generalized plane stress.

For antiplane problems, the complex form solution can be written in terms of a single stress function \( \varphi(z) \) as

\[
\sigma_{31} - i \sigma_{32} = \psi''(z), \quad \mu_{31} = \text{Re} \left[ \psi(z) \right].
\]

(2.4)

2.2 Lekhnitskii Formulation

For two-dimensional anisotropic elasticity, there are two different formulations to the literature. One is the Lekhnitskii approach [2,13] which starts with the equilibrium stress functions followed by compatibility equations; the other is the Stroh formalism [15,16] which starts with the compatible displacement functions followed by equilibrium equations. Since both of them deal with the same basic equations given in (2.1), they should be equivalent. The equivalence of these two formulations has been discussed by Suo [17] and will also be discussed in the next subsection. If we ignore the body forces and the particular solutions of the nonhomogeneous system, the general solutions provided in the Lekhnitskii formulation may be written as follows.

\[
\sigma_{31} = 2 \text{Re} \left[ \mu_1 \Phi(\zeta(z)) + \mu_2 \Phi'(\zeta(z)) + \mu \Phi'(\zeta(z)) \right],
\]

\[
\sigma_{22} = 2 \text{Re} \left[ \Phi_{1}(z) + \Phi_{2}(z) + \mu_1 \Phi_{1}\Phi_{2}(z) \right],
\]

\[
\sigma_{21} = -2 \text{Re} \left[ \mu_1 \Phi_{1}(z) + \mu_2 \Phi_{2}(z) + \mu_1 \Phi_{1}\Phi_{2}(z) \right],
\]

\[
\sigma_{12} = 2 \text{Re} \left[ \mu_1 \Phi_{1}(z) + \mu_2 \Phi_{2}(z) + \mu_1 \Phi_{1}\Phi_{2}(z) \right],
\]

\[
\sigma_{21} = 2 \text{Re} \left[ \mu_1 \Phi_{1}(z) + \mu_2 \Phi_{2}(z) + \mu_{12}\Phi_{1}\Phi_{2}(z) \right],
\]

(2.5a)
\begin{align}
  u_1 &= 2 \text{Re} \sum_{\gamma=1}^{3} a_{\gamma a} \Phi_{\gamma}(z_a), \\
  u_2 &= 2 \text{Re} \sum_{\gamma=1}^{3} a_{\gamma a} \Phi_{\gamma}(z_a), \\
  u_3 &= 2 \text{Re} \sum_{\gamma=1}^{3} a_{\gamma a} \Phi_{\gamma}(z_a). \tag{2.5b}
\end{align}

In the above, \( \Phi_{\gamma}(z_a), \alpha = 1, 2, 3, \) are three holomorphic functions of complex variables \( z_a = (a_1 + p_a)z_a \), which will be determined by the boundary conditions, \( a_{\gamma a} \) and \( a_{\gamma a} \) are defined as
\begin{align}
  \eta_1 &= -\varepsilon_1(p_\alpha) / \varepsilon_1(p_\alpha), \\
  \eta_2 &= -\varepsilon_2(p_\alpha) / \varepsilon_2(p_\alpha), \\
  \eta_3 &= -\varepsilon_3(p_\alpha) / \varepsilon_3(p_\alpha), \\
  \xi_1(p_\alpha) &= \hat{S}_{11} p_\alpha - 2 \hat{S}_{12} p_\alpha + \hat{S}_{13}, \\
  \xi_2(p_\alpha) &= \hat{S}_{12} p_\alpha - (\hat{S}_{13} + \hat{S}_{13}) p_\alpha + \hat{S}_{14}, \\
  \xi_3(p_\alpha) &= \hat{S}_{13} p_\alpha - 2 \hat{S}_{14} p_\alpha + (2 \hat{S}_{12} + \hat{S}_{13}) p_\alpha - 2 \hat{S}_{23} p_\alpha + \hat{S}_{22}. \tag{2.6a}
\end{align}

and
\begin{align}
  a_{\gamma a} &= \hat{S}_{11} p_\alpha + \hat{S}_{12} - \hat{S}_{11} p_\alpha + \hat{S}_{12} - \hat{S}_{13} p_\alpha - \hat{S}_{13}, \\
  a_{\gamma a} &= \hat{S}_{12} p_\alpha + \hat{S}_{22} - \hat{S}_{12} - \hat{S}_{14} p_\alpha + \hat{S}_{14}, \\
  a_{\gamma a} &= \hat{S}_{14} p_\alpha + \hat{S}_{23} - \hat{S}_{14} - \hat{S}_{15} p_\alpha + \hat{S}_{15}, \alpha = 1, 2, 3, \\
  a_{\gamma a} &= \eta_1 (\hat{S}_{11} p_\alpha + \hat{S}_{12} - \hat{S}_{13} p_\alpha - \hat{S}_{13}), \\
  a_{\gamma a} &= \eta_2 (\hat{S}_{12} p_\alpha + \hat{S}_{22} - \hat{S}_{12} - \hat{S}_{14} p_\alpha + \hat{S}_{14}), \\
  a_{\gamma a} &= \eta_3 (\hat{S}_{14} p_\alpha + \hat{S}_{23} - \hat{S}_{14} - \hat{S}_{15} p_\alpha + \hat{S}_{15}). \tag{2.6b}
\end{align}

\( p_\alpha, \alpha = 1, 2, 3, \) are the complex parameters with positive real parts, which characterize the degree of anisotropy and may be determined by the following sixth order characteristic equation
\begin{equation}
  \varepsilon_1(p_\alpha) \varepsilon_2(p_\alpha) - (\varepsilon_3(p_\alpha))^2 = 0. \tag{2.7}
\end{equation}

Note that the general solutions given in (2.5) are only valid for \( p_\alpha \) distinct. In the case of repeated root, modified solutions should be used which will be discussed in the next section.

By viewing the component form solutions shown in (2.5) by Lehmkuski formulation and the matrix form solutions shown in (2.8) by Stroh formalism, it can easily be seen that

\[ \alpha f_\alpha(x_\alpha) = \Phi_\alpha(x_\alpha), \quad \alpha \neq \text{summed} \quad (2.12) \]

and

\[ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \]

\[ B = \begin{pmatrix} -p_1 c_1 & -p_2 c_2 & -p_3 c_3 \\ c_1 & -p_1 c_1 & -p_2 c_2 \\ -p_3 c_3 & c_2 & -p_1 c_1 \end{pmatrix}, \quad (2.13a) \]

where \( n_\alpha \) and \( a_\alpha \) are given in (2.6), and \( c_\alpha, \alpha = 1, 2, 3 \), are the normalization factors which can be determined from (2.11) as

\[ 2c_1^2 = (a_{11} + a_{22} - n_1 n_2)^{-1}, \]

\[ 2c_2^2 = (a_{22} + a_{33} - n_2 n_3)^{-1}, \]

\[ 2c_3^2 = (a_{33} + a_{11} - n_3 n_1)^{-1}. \quad (2.23b) \]

The expression for the material eigenvector matrices \( A \) and \( B \) shown in (2.13) has been proved numerically to be identical to those obtained from (2.10) and normalized by (2.11). The material eigenvalues \( p_\alpha \) obtained from (2.10a) has also been proved numerically to be identical to those found by (2.7).

It should be noted that the main difference between Lehmkuski and Stroh formulations attributes to the introduction of the eigenrelation given in (2.10) and the normalization given in (2.11). With this eigenrelation and normalization, many problems which are left with unsolved linear algebraic system can be solved explicitly. Moreover, many complex variable form solutions may be transformed to real form solutions through the use of some identities which relate real quantities such as \( \Phi \) to complex quantities such as \( A \) and \( B \) by the eigenrelation shown in (2.10).

Due to the special feature mentioned above, in the following discussion we will use Stroh formalism to complete our derivation from anisotropy to isotropy. Using the relations given in (2.12) and (2.13), all the results obtained can also be applied to the solutions found by the Lehmkuski approach.

3. Deduction from Anisotropic to Isotropic Elasticity

The general solutions to the basic equations (2.1) of anisotropic elasticity obtained through the Lehmkuski formulation or Stroh formalism are usually presented (as in the last section) for the case when the material eigenvalues \( p_\alpha \) are distinct. They are not valid for isotropic materials since their material eigenvalues \( p_\alpha \neq \pm 1 \) are repeated. For the materials with repeated eigenvalues and eigenvectors (called degenerate materials), say \( p_1 = p_2 \) and \( \xi = \xi_2 \), a general reduction principle was provided by Chadwick and Smith [11] and Ting [18]. To connect the formalism of anisotropic materials and degenerate materials smoothly, a modified secant formalism for almost degenerate materials \( p_\alpha \) (almost repeated) has been suggested by Ting and Hwu [20]. Although all these reduction principles have been presented, to the author’s knowledge, no detail derivation has been done to connect directly the stress functions between the anisotropic and isotropic elasticity, i.e., \( \psi \) in Muskhelishvili formulation, and \( f_\alpha(x_\alpha) \) in Stroh formalism which also relate to \( \Phi_\alpha(x_\alpha) \) in Lehmkuski formulation by (2.12). In this section, the derivation will be provided in the manner of anisotropic materials to almost degenerate materials to degenerate materials to isotropic materials. In the Appendix, a simpler derivation is given for the special cases that \( f_1(x_1) \) and \( f_2(x_1) \) have the same function form (i.e., \( f_1 = f_2 = f \)) and the in-plane and anti-plane components are assumed to be decoupled so that \( f_3 \) can be disregarded during deduction from anisotropic elasticity to isotropic elasticity.

3.1 Anisotropic Materials

To construct the general solutions for the almost degenerate materials smoothly, we replace the stress functions \( f_\alpha(x_\alpha) \) in (2.8b) by \( \sum_{k=1}^{n} \tilde{b}_k \tilde{f}_k(x_\alpha) \), or in matrix form,

\[ \tilde{f}(x) = \sum_{k=1}^{n} \tilde{b}_k \tilde{f}_k(x), \quad (3.1) \]

where \( \tilde{f}_k \) is a \( 3 \times 1 \) vector containing the coefficients \( \tilde{b}_k \), and the angular bracket \( \langle \cdot \rangle \) stands for the diagonal matrix, i.e., \( \langle f_\alpha(x_\alpha) \rangle = \delta_{\alpha \beta} f_\alpha(x_\alpha) \) in which each component is varied according to the Greek index \( \alpha \). With this replacement, the general solutions may now be rewritten as

\[ \tilde{a} = 2 \sum_{k=1}^{n} \Re \{ \tilde{b} \tilde{F}_k(x) \tilde{g}_k \}, \quad (3.2a) \]

\[ \tilde{f}(x) = 2 \sum_{k=1}^{n} \Re \{ \tilde{b} \tilde{F}_k(x) \tilde{g}_k \}. \]

In this new representation, the components of each vector term, e.g., the \( k \)th term of the series contain the same function \( f_k \) with different argument
3.2 Almost Degenerate Materials

Since the eigen-vector matrices $\mathbf{A}$ and $\mathbf{B}$ will become almost singular for almost degenerate materials, the following replacement has been suggested by Ting and Elwa [20]:

$$
\mathbf{A} = \mathbf{A}^* \mathbf{E}^{-1}, \quad \mathbf{B} = \mathbf{B}^* \mathbf{E}^{-1},
$$

(3.3a)

in which $\mathbf{A}^*$ and $\mathbf{B}^*$ will not be singular for (almost) degenerate materials. For the case of $p_1 \neq p_2$,

$$
\mathbf{A}^* = \begin{bmatrix} a_1 & a_2 \gamma & a_3 \gamma \end{bmatrix}, \quad \mathbf{B}^* = \begin{bmatrix} b_1 \gamma & b_2 & b_3 \gamma \end{bmatrix},
$$

(3.3b)

$$
\mathbf{E} = \begin{bmatrix} \gamma & 1 & 0 \\ 0 & \gamma & 1 \\ 0 & 0 & 0 \end{bmatrix},
$$

(3.3c)

and

$$
\gamma^2 = p_1 - p_2.
$$

(3.3d)

In the above, the independent eigenvectors $(\xi_2, \xi_3)$ and $(\xi_3, \xi_1)$ are determined from the eigenrelation shown in (2.19), and the generalized eigenvector $(\xi_3, \xi_1)$ are obtained by differentiating (2.10) with respect to $\xi_i$, i.e.,

$$
\xi_2 \xi_1 = p_1 \xi_1 + \xi_2, \quad \xi_3 \xi_1 = \xi_3 \xi_2.
$$

(3.4)

Similar to the eigenvector matrices $\mathbf{A}$ and $\mathbf{B}$, the generalized eigen-vector matrices $\mathbf{A}^*$ and $\mathbf{B}^*$ should also be normalized in order to take advantage of some identities developed in Stroh formalism. The normalization $\mathbf{A}$ performed according to

$$
\mathbf{A}^* \mathbf{Y} + \mathbf{B}^* \mathbf{Y} = \mathbf{Y},
$$

(3.5a)

where

$$
\mathbf{Y} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & p_1 - p_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

(3.5b)

By employing the replacement given in (3.3a) and letting $q_2 = \theta q_2'$ in (3.2), the modified general solutions for materials with $p_1 \neq p_2$ are

$$
\gamma = 2 \sum_{k=1}^{n} \text{Re} \left( \mathbf{A}^* \mathbf{E}_k \mathbf{y}_k \right),
$$

$$
\phi = 2 \sum_{k=1}^{n} \text{Re} \left( \mathbf{B}^* \mathbf{E}_k \mathbf{y}_k \right)
$$

(3.6a)

where

$$
\mathbf{E}_k(x) = \mathbf{E}^{-1} \mathbf{E}_k(x).
$$

(3.6b)


2.3 Degenerate Materials

By using the general solution shown in (3.6) for almost degenerate materials, and replacing $\sum_{k=1}^{n} \text{Re} \left( \mathbf{E}_k(x) \right)$ by $\mathbf{f}(x)$, the solution for the degenerate materials with $p_1 = p_2$ may now be simplified to

$$
\gamma = 2 \text{Re} \left( \mathbf{A}^* \mathbf{f}(x) \right), \quad \phi = 2 \text{Re} \left( \mathbf{B}^* \mathbf{f}(x) \right),
$$

(3.7a)

where

$$
\mathbf{f}(x) = \mathbf{E}^{-1} \mathbf{f}(x) = \begin{bmatrix} f_1(x_1) + \frac{\theta f_2(x_2)}{\theta} & f_2(x_1) \\ f_2(x_1) & f_3(x_1) \end{bmatrix}.
$$

(3.7b)

It should be noted that $\mathbf{f}(x)$ may not be a holomorphic function vector since $\frac{\theta f_2(x_2)}{\theta}$ is in general not holomorphic. Moreover, it is hard to imagine how $\mathbf{f}(x)$ be calculated from $\mathbf{E}^{-1} \mathbf{f}(x)$ where $\mathbf{f}(x)$ is holomorphic and $\mathbf{E}^{-1}$, by (3.3c), is singular when $p_1 = p_2$. The detailed calculation of $\mathbf{f}(x)$ can be found in the examples shown in this paper.

3.4 Isotropic Materials

Although the eigenvalues $p_1 = p_2 = p_3 = 1$ for isotropic materials appear to be the right, two independent eigenvectors can be found from this triple root. Therefore, the cases of isotropic materials should be treated as double root problems, and the solutions shown in (3.7) can be applied directly. By the procedure stated from (3.3b) to (3.5b), the generalized eigen-vector matrices $\mathbf{A}^*$ and $\mathbf{B}^*$ for isotropic materials are found to be [20]

$$
\mathbf{A}^* = \begin{bmatrix} k_1 & -i k_2 & 0 \\ i k_2 & k_1 & 0 \\ 0 & 0 & \mu \end{bmatrix},
$$

$$
\mathbf{B}^* = \begin{bmatrix} \frac{2k_2 k_1}{\mu} & k_1 & 0 \\ -2k_2 & -i k_1 & 0 \\ 0 & 0 & k_3 \end{bmatrix},
$$

(3.8)

where $k_2 = 1/(2\mu(1+k))$ and $k_3 = -i/2\mu$. Substituting (3.8) into (3.7), the general solutions for isotropic materials can now be simplified to

$$
\gamma = 2 \text{Re} \left[ \begin{bmatrix} k_1 f_1(x_1) + f_2(x_1, x_2) - \frac{i k_2}{\mu} f_3(x_1) \\ k_1 f_1(x_1) + f_2(x_1, x_2) - \frac{i k_2}{\mu} f_3(x_1) \\ k_3 f_2(x_1) \end{bmatrix} \right],
$$

(3.9a)
and

\[ \dot{\phi} = 2 \Re \left\{ k_1 f''(z) + f''(z, \bar{z}) + k_1 f''(z) \right\} \]

where

\[ f''(z, \bar{z}) = \frac{\partial^2 f(z)}{\partial z^2} \]  

(3.9c)

Note that we use \( f''(z, \bar{z}) \) instead of \( f''(z) \) to represent \( \frac{\partial^2 f(z)}{\partial z^2} \) since it is not only a function of \( z \) but also a function of \( \bar{z} \), which can easily be seen by \( \frac{\partial f(z)}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}} f(z) = f''(z) \).

4. CORRESPONDENCE RELATION

In order to find the correspondence relations between anisotropic and isotropic elasticity, we reorganize the solutions shown in (3.9) into the form presented in (2.3) for Muskhelishvili formulation. To this end, we first express \( f''(z, \bar{z}) \) in terms of some holomorphic functions in order to have a direct comparison with the holomorphic functions \( \phi(z) \) and \( \psi(z) \). By applying the chain rule for differentiation in (3.9c), we have

\[ f''(z, \bar{z}) = \frac{\partial^2 f(z)}{\partial z^2} + \frac{\partial}{\partial \bar{z}} \left( \frac{\partial f(z)}{\partial z} \right) \]

(4.1a)

where both \( f''(z) \) and \( \frac{\partial f(z)}{\partial z} \) are holomorphic functions, and are related to \( f''(z) \) by

\[ f''(z) = \frac{\partial^2 f(z)}{\partial z^2}, \quad \frac{\partial f(z)}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}} f(z) \]

(4.1b)

One should be very careful about the definitions of \( f''(z) \) and \( \frac{\partial f(z)}{\partial z} \) which are obtained by treating \( z \) and \( \bar{z} \) as totally independent variables, although \( z = x + iy \) and \( \bar{z} = x - iy \). For example, if \( f''(z) = 2z \), then \( f''(x, y) = 2x + i = -i(z - \bar{z}) + 1 \).

The calculation of the stresses \( \sigma_{ij} \) may now be performed by applying (2.9), (3.9b) and (4.1), and noting that

\[ \dot{\phi} = \frac{\partial \phi}{\partial x} + i \frac{\partial \phi}{\partial y} \]

(4.2)

Organizing the results for the stresses and that shown in (3.9a) for the displacements into the form of Muskhelishvili formulation, we have

\[ \sigma_{11} + \sigma_{22} = -2k_1 \Re \{ \phi(z) \} \]

\[ \sigma_{22} - \sigma_{11} + 2ic_1 = -4k_1 \Re \{ \phi(z) \} + f''(z) + \tau \frac{\partial f''(z)}{\partial \bar{z}} - \frac{\tau^2}{2} f''(z) \]

\[ \sigma_{11} - i\sigma_{22} = 2k_1 \phi(z) \]

\[ u_1 + iu_2 = \]

\[ 2k_1 \left( \frac{f''(z)}{2} + \frac{f''(z) + \tau \frac{\partial f''(z)}{\partial \bar{z}} - \frac{\tau^2}{2} f''(z)}{2} \right) - \frac{\tau^2}{2} f''(z) \]

\[ u_3 = 2 \Re \{ k_1 \phi(z) \} \]

(4.3)

By comparing the above equations with (2.3) and (2.4), the corresponding relations may be written as

\[ \phi(z) = -2k_1 \phi(z) \]

\[ \psi(z) = -4k_1 \left( \phi(z) + f''(z) - \frac{\tau^2}{2} f''(z) \right) \]

(4.4a)

where \( f''(z) \) is defined in (4.1b), which is related to \( f''(z) \), and \( f''(z) \), \( \alpha = 1, 2, 3 \), are related to \( f'(z) \) of anisotropic problems by

\[ f''(z) = f''(z) \alpha \]

(4.4b)

and by (3.7b)

\[ f''(z) = \left\{ \begin{array}{ll}
\frac{f''(z)}{2} + \frac{\phi(z)}{2} & \text{if } \alpha = 1 \\
\frac{f''(z)}{2} - \frac{\phi(z)}{2} & \text{if } \alpha = 2 \\
 f''(z) & \text{if } \alpha = 3
\end{array} \right. \]

(4.4c)

With the correspondence relations provided in (4.4), we now summarize the reduction procedures from anisotropic to isotropic problems as follows.

(1) With \( f'(z) \) of anisotropic elasticity problems found by Stroh formalism or Lekhnitskii formulation (\( f'(z) = f'(z) \alpha \)), \( f'(z) \) can be calculated through the use of (4.4c).

(2) Calculate \( f''(z) \) by (4.4b) and \( f''(z) \) by (4.4b).

(3) Calculate \( \phi(z), \psi(z) \) and \( \psi(z) \) by (4.4a).

As we said in the introduction of this paper, the present correspondence relations provide a simple way to reduce the anisotropic solutions to the corresponding isotropic solutions. Hence, any newly found anisotropic solutions can easily be checked analytically (without numerical calculation) by the existing
corresponding isotropic solutions. Moreover, from (4.4a) and (4.4c), we see that

$$f_s(x) = \phi(x)$$

This information may suggest an easier way to solve the anisotropic problems by referring to the corresponding isotropic solutions.

5. EXAMPLES

In order to demonstrate the usefulness of the present correspondence relations, four typical examples concerning holes, cracks, line forces or dislocations, and punch indentation are illustrated explicitly in this section.

5.1 Holes

Consider a traction-free elliptical hole with semi-axes $a$ and $b$ in an unbounded anisotropic medium subject to a uniform loading at infinity. The solution to this problem has been provided by Hwu [7] through the use of Stroh formalism, which is

$$g = \mathbf{g}^m = \text{Re}(\mathbf{g}^m) = \mathbf{g}^m = \text{Re}(\mathbf{g}^m)$$

and

$$\mathbf{z}_m = \mathbf{z}_m + \sqrt{\frac{a^2 + \mathbf{p}^2 b^2}{a^2}} \mathbf{z}_m$$

(5.1c)

In (5.1), $\mathbf{T}$ and $\mathbf{B}$ are defined by (2.10d), whereas

$$\mathbf{f}_m = \begin{pmatrix} \mathbf{f}_m \\ \mathbf{f}_m \end{pmatrix}, \quad \mathbf{f}_m = \begin{pmatrix} \mathbf{f}_m \\ \mathbf{f}_m \end{pmatrix}$$

and

$$\mathbf{f}_m = \begin{pmatrix} \mathbf{f}_m \\ \mathbf{f}_m \end{pmatrix}, \quad \mathbf{f}_m = \begin{pmatrix} \mathbf{f}_m \\ \mathbf{f}_m \end{pmatrix}$$

(5.2)

For the problems when loading is applied at infinity, $\mathbf{f}_m$ are given and $\mathbf{f}_m$ can be determined by using the stress-strain relation. In the cases of unidirectional tension $\sigma$ which makes an angle $\alpha$ with the positive $x_1$-axis, i.e., $\sigma_{11} = \sigma \cos^2 \alpha$, $\sigma_{22} = \sigma \sin^2 \alpha$, $\sigma_{12} = \sigma \cos \alpha \sin \alpha$ and all the other stress components are zero, we have for the plane strain condition [7]

$$\mathbf{f}_m = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}, \quad \mathbf{f}_m = \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix}$$

(5.3a)

where

$$\mathbf{g}(\alpha) = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}, \quad \mathbf{B}_0 = \begin{pmatrix} \frac{1}{2} \mathbf{B}_0 \\ \frac{1}{2} \mathbf{B}_0 \end{pmatrix}, \quad \mathbf{B}_0 = \begin{pmatrix} \frac{1}{2} \mathbf{B}_0 \\ \frac{1}{2} \mathbf{B}_0 \end{pmatrix}$$

(5.3b)

By comparing (5.1) with the general solution given in (2.8a), we have

$$f(z) = \mathbf{g}(z) \mathbf{B}_0 + \mathbf{g}(z) \mathbf{B}_0$$

(5.4)

In order to find $f(z)$, we use (4.4c) which leads to

$$f(z) = \mathbf{g}^m = \mathbf{B}_0 = \mathbf{B}_0$$

(5.5)

where

$$\mathbf{g} = \mathbf{Y}(\mathbf{a}^2 + \mathbf{p}^2 b^2) \mathbf{z}_m$$

(5.5b)

Note that in the above derivation we have used the relation given in (3.3a) and [20]

$$E - 1 = \mathbf{g}^m$$

(5.6)

With all the information listed in (3.3c), (3.5b), (3.6), (2.10d) and (5.3b), $f(z)$ can be found explicitly for isotropic materials. With the results of $f(z)$, employment of (4.4b) and (4.1b) will lead to

$$f(z) = \frac{\sigma}{16\mu_0} \left[ (2e^{-mz} - 1) - 2(me^{-mz} - 1) \right]$$

(5.7a)

and

$$f(z) = \frac{m}{8\mu_0} \left[ 1 + 2(e^{mz} - m)R \right]$$

(5.7b)

where

$$m = \frac{a - b}{a + b}$$

(5.7b)

$R = \frac{a + b}{2}$.  

Applying the correspondence relation given in (5.4a), we obtain

\[ \psi(z) = \frac{R_0}{4} \left[ \frac{1}{z} + (2e^{2\alpha} - m) \frac{1}{z^3} \right], \]

\[ \psi(z) = -\frac{R_0}{4} \left[ 2z^2 \frac{z^6 - \alpha^2}{z^2 - \alpha^2} \right], \]

\[ \omega(z) = 0, \]  
(5.8)

which are identical to those shown in England [3].

If we consider a special case of circular hole \((b = a)\) subject to a unidirectional tension parallel to the \(x_1\) axis at infinity, i.e., \(m = 0\) and \(\alpha = 0\), eq.(5.8) will be reduced to

\[ \psi(z) = \frac{\sigma}{4} \left( z + \frac{2\alpha^2}{z} \right), \]

\[ \psi(z) = -\frac{\sigma}{4} \left( z + \frac{a^2}{z} \right), \]  
(5.9)

which are identical to those shown in Muskhelishvili [14].

5.2 Cracks

An elliptical hole can be made into a crack of length \(2b\) by letting \(b\) be equal to zero. The solution given in (5.1) is then applicable to the crack problems with \(b = 0\). If a unidirectional tension parallel to \(x_1\)-axis is applied at infinity, that is \(\alpha = \frac{a}{2}\). By (5.8) with \(m = 1\) and \(\alpha = \frac{a}{2}\), we obtain

\[ \psi(z) = \frac{\sigma}{4} \left( z + \frac{\alpha^2}{z} + \sqrt{z^2 - \alpha^2} \right), \]

\[ \psi(z) = \frac{\sigma}{4} \left( z + \frac{\alpha^2}{z^2 \sqrt{z^2 - \alpha^2}} \right), \]  
(5.10)

which are identical to those shown in Muskhelishvili [14].

5.3 Line forces and dislocations

Consider an isolated singularity in an infinite homogeneous anisotropic medium. If the singularity is a line force \(f\) and displacement \(\hat{b}\) located at the point \((\xi_1, \xi_2)\), the complex function vector \(f(z)\) associated with this problem may be written as [10]

\[ f(z) = \frac{1}{2\pi i} \sum_{\xi_1} \frac{1}{z - \xi_1} \approx \left( \frac{\hat{b}^T \hat{b}}{\beta} \right). \]  
(5.11)

With \(f(z)\) given in (5.11), by the procedure stated in Section 4, which has also illustrated for the

hole and crack problems, we obtain

\[ f'(z) = \frac{1}{2\pi i} \left[ \frac{1}{z - \xi_1} \right], \]

\[ f(z) = \frac{1}{2\pi i} \left[ \frac{1}{z - \xi_1} \right], \]

\[ f'(z) = \frac{1}{2\pi i} \left[ \frac{1}{z - \xi_1} \right], \]

\[ f(z) = \frac{1}{2\pi i} \left[ \frac{1}{z - \xi_1} \right], \]  
(5.11)

and the correspondence relation (4.4a) leads to

\[ \psi(z) = \frac{\sigma}{4} \left( z + \frac{\alpha^2}{z} \right), \]

\[ \psi(z) = \frac{\sigma}{4} \left( z + \frac{\alpha^2}{z^2 \sqrt{z^2 - \alpha^2}} \right), \]  
(5.11)

Again, the solution found in (5.13) are identical to those shown in Muskhelishvili [14].

5.4 Punch Indentation

Consider the problems that a set of rigid punches of given profiles are brought into contact with the surface of a half-plane and are allowed to indent the surface in such a way that the punches completely adhere to the half-plane on initial contact and during the subsequent indentation no slip occurs and the contact region does not change. The general solutions for an anisotropic elastic half-plane can be found in Fan and Hsu [4]. The corresponding solutions for isotropic media can be found in Muskhelishvili [14]. To show the validity of the present correspondence relations, we examine a special case of indentation by a single punch with a flat-ended profile which makes contact with the half-plane over the region \(|x| \leq a\), and a downward force \(q = 0\) at \((0, P, 0)\) applied on the punch is given. The solutions for anisotropic media are [4]

\[ f'(z) = \frac{1}{2\pi i} \left[ \frac{1}{z - \xi_1} \right], \]

\[ f(z) = \frac{1}{2\pi i} \left[ \frac{1}{z - \xi_1} \right], \]  
(5.14a)
where

\[ L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \]

\[ L_4 = \frac{1}{\nu} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \]  

(5.14b)

\[ \varepsilon_{11} \left( -v_{13} - \frac{1}{2} \right) \text{ and } \Lambda = \left[ \Lambda_1, \Lambda_2, \Lambda_3 \right] \text{ can be deter-

\[ (M^{-1} + e_{13} a_{13}^{-1}) \Lambda = 0, \quad M = -i \Lambda A^{-1}. \]  

(5.14c)

Moreover, \( \Delta \) is normalized by

\[ \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{\frac{2 \lambda + 3 \mu}{3 \lambda + 2 \mu}} \end{bmatrix} \Delta = L. \]  

(5.14d)

The explicit solution for \( \Delta \) can be found by a way similar to those described in [8]. For the ease of reference, the explicit solution for \( \Delta \) in the case of isotropic media is given below:

\[ \Delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{\frac{2 \lambda + 3 \mu}{3 \lambda + 2 \mu}} \end{bmatrix}. \]

With \( f'(z) \) given by (5.14), in a similar way to that of examples 5.1–5.3, we obtain

\[ f'_1(z) = \frac{-i \nu}{8 \pi \mu_1} X(z), \quad f'_2(z) = \frac{-i \nu}{4 \pi \mu_1} X(z), \]

\[ f'_3(z) = f'_4(z) = 0, \]  

(5.15a)

where

\[ X(z) = (z - a)^{-1+i\sigma} - (z - a)^{-1-i\sigma}. \]  

(5.15b)

Through the use of the correspondence relation given in (4.4a), we obtain

\[ \psi(z) = \frac{i \nu}{2 \pi} X(z), \]

\[ \psi'(z) = \frac{i \nu}{2 \pi} \left[ X(z) - X(z) - \sigma X'(z) \right], \]

\[ \omega'(z) = 0, \]  

(5.16)

which are identical to those shown in Muskhelishvili [14].

6. CONCLUDING REMARKS

A simple correspondence relation between the stress functions of anisotropic and isotropic elasticity is established in Eq. (4.4). With this relation, a reduction procedure from anisotropy to isotropy is stated in the paragraph following Eq. (4.4c). Thus, any new found anisotropic solutions can easily be checked analytically (without numerical calculation) by the existing corresponding isotropic solutions. Moreover, an easier way to solve anisotropic problems by referring to the corresponding isotropic solutions is suggested in Eq. (4.5). Hence, the unsolved anisotropic problems may be solved by using semi-inverse method with reference to the stress function \( \psi(z) \) (or \( \omega(z) \) of the antiplane shear problems) of the corresponding isotropic problems. In other words, to solve an unsolved anisotropic problem, we first look for the existing solutions of its corresponding isotropic problem. Then we assume the stress functions \( f_0(z_0) \) (or \( \Phi(z_0) \) in Lekhnitskii formulation) of the anisotropic problems have the same function form as \( \psi(z) \) (or \( \omega(z) \) in antiplane shear problems) of its corresponding isotropic problems. By this way, the only unknowns remaining to be determined are the coefficients associated with the assumed stress functions \( f_0(z_0) \), which may then be found by satisfying the boundary conditions set for the problems.

To have a clear picture, the results of the examples discussed in the previous section are summarized in Table 1. From Table 1, we see that the function form of \( f_0(z_0) \) is closely related to that of \( \psi(z) \). On the other hand, due to the relatively complicated relation between \( \psi(z) \) and \( f_0(z_0) \) shown in (4.4a), the function form of \( \psi(z) \) is usually very different from that of \( f_0(z_0) \), which can also be seen from the results of the previous section.

Although a direct approach from \( f_0(z_0) \) to \( \psi(z) \) has been established in this paper and the resemblance between \( \psi(z) \) and \( f_0(z_0) \) is obvious, it is still difficult to construct a direct approach from \( \psi(z) \) to \( f_0(z_0) \). For example, by Table 1 we have \( \psi(z) = \frac{1}{2} (z + a)^{1+i\sigma} \) for the circular hole problems, how can we know in its corresponding anisotropic stress functions are \( f_0(z_0) = 0 + d_0 \sigma^0 \) instead of \( f_0(z_0) = a_0 + d_0 \sigma^0 \). Therefore, when one tries to solve an unsolved anisotropic problem, one may use its corresponding \( \psi(z) \) as a good reference but not stick to its original form. A correct form may be found by combining the information provided by the method of analytical continuation and the technique of conformal mapping.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Comparison of the stress functions between anisotropic and isotropic elasticity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_0(x)$, $\sigma_1(x)$</td>
</tr>
<tr>
<td>elliptic hole</td>
<td>$\sigma_{11} + 4\sigma_{12} + \sigma_{22}$, eq (5.6)</td>
</tr>
<tr>
<td>circular hole</td>
<td>$\sigma_{11} + 2\sigma_{12} + \sigma_{22}$, eq (5.6)</td>
</tr>
<tr>
<td>Crack</td>
<td>$\sigma_{11} + 2\sigma_{12} + \sigma_{22}$, eq (5.6)</td>
</tr>
<tr>
<td>Line force (Dislocation)</td>
<td>$\sigma_{00}(x - l_n)$, eq (5.13)</td>
</tr>
<tr>
<td>Punch indentation</td>
<td>$\int -\frac{1}{4\pi l_n} \left(\frac{m^2}{r^2} - 1\right) d^2 r$, eq (5.14)</td>
</tr>
</tbody>
</table>

Remarks:
- $\psi(x) = -S\phi(x)$, eq (4.4a).
- $\psi(x) = \frac{1}{2}(\sigma_{11} + \sigma_{22})$, $n = 1, 2, 3$, eq (4.4b).
- $\frac{1}{2}(\sigma_{11} + \sigma_{22})$, $n = 1, 2, 3$, eq (4.4a).
- $\psi(x) = \int \frac{1}{2\pi l_n} \left(\frac{m^2}{r^2} - 1\right) d^2 r$, eq (5.15).

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REFERENCES

APPENDIX

If we consider the special cases that \( f_{1}(s) \) and \( f_{2}(s) \) have the same function form (i.e., \( f_{1} = f_{2} = f \)), and \( f_{3} \) can be disregarded. The stress function \( \phi \) provided in (2.8a) may be written as

\[
\phi = 2 \text{Re} \left\{ f(z) \left[ \begin{array}{cc} 1 & -p_{0}^{2} \alpha \omega_{3} \\ p_{0} \alpha \omega_{3} & 1 \end{array} \right] \right\}, \quad \alpha = 1, 2, \quad (A1)
\]

where \( B_{\alpha} \) is a 2x2 matrix obtained from (2.13a), by deleting the 3rd row and 3rd column of \( B \), i.e.,

\[
B_{\alpha} = \left[ \begin{array}{cc} c_{1} & -p_{0}c_{2} \\ -p_{0}c_{1} & c_{2} \end{array} \right]. \quad (A2)
\]

\( h \) is a 2x1 column matrix which contains two complex numbers \( h_{1} \) and \( h_{2} \), i.e.,

\[
h = \left[ \begin{array}{c} h_{1} \\ h_{2} \end{array} \right]. \quad (A3)
\]

By direct calculation of (A1) using (A2) and (A3), we have

\[
\left\{ \begin{array}{l}
\phi_{1} \\
\phi_{2}
\end{array} \right\} = 2 \text{Re} \left\{ \begin{array}{l}
f(z)h_{1} + f(\bar{z})h_{2} \\
f(\bar{z})h_{1} - f(z)h_{2}
\end{array} \right\}, \quad (A4)
\]

For isotropic materials, we substitute \( p_{1} = p_{2} = i \) into (A4), and obtain

\[
\left\{ \begin{array}{l}
\phi_{1} \\
\phi_{2}
\end{array} \right\} = 2 \text{Re} \left\{ \begin{array}{l}
f'(z)h_{1} + f'(\bar{z})h_{2} \\
f'(\bar{z})h_{1} - f'(z)h_{2}
\end{array} \right\}, \quad (A5)
\]

where \( f'(z) = \frac{df}{dz}, \quad f'(\bar{z}) = \frac{df}{d\bar{z}} \), and \( h = h_{1} + ih_{2} \). The Muskhelishvili’s representation for the traction results is

\[
\phi_{1} + i \phi_{2} = \left\{ \chi(z) + z \psi(z) + \psi(\bar{z}) \right\}. \quad (A6)
\]

In order to get the correspondence relations between anisotropic and isotropic elasticity, we now reconstruct the results shown in (A5) into the form of (A6). They are

\[
\phi_{1} + i \phi_{2} = \bar{h}(f(z) - \bar{h}f'(z)) + h^{2}f'(z) + h^{2}\left( f'(z) + 2\bar{h}f(z) \right), \quad (A7)
\]

where \( \bar{h} = h - ih \). By comparison of (A6) and (A7), we get

\[
\psi(z) = -ih\psi(z), \quad (A8)
\]

\[
\psi(z) = \left\{ \bar{h}f'(z) + h^{2}\left( f'(z) + 2\bar{h}f(z) \right) \right\}. \quad (A8)
\]

異向性與等向性彈性力學問之對等關係

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摘要
在二維異向性彈性力學的理論基礎上，Muskheleishvili 為異向性材料及 Stroh 為異向性材料所發展出來之複變理論中相當有名。本文即是利用此二理論推導異向性與等向性彈性力學間的一些關係式，經由這些等向關係，任一個異向性之異向性彈性力學解可直接由等向解推算而得。本文現有之等向性解和異向性解之數值計算。同時，因異向性材料使異向性彈性力學解有著異向性彈性力學解的特徵，這些關係式也將提供一種處理異向性彈性力學方法。

關鍵詞：異向性彈性力學，等向性彈性力學，複變數法，平面問題，對等關係。

* 教性

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493