Free Vibration of Delaminated Composite Sandwich Beams

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Consider a sandwich plate with anisotropic composite laminated faces and an ideally orthotropic honeycomb core. In this paper, a one-dimensional model considering the transverse shear effect and rotary inertia for the free vibration analysis of a sandwich plate with an across-the-width delamination located at the interface between the upper face and core is developed. With this model, the natural frequencies and mode shapes of the delaminated composite sandwich beams can be obtained by solving the eigenvalues and eigenvectors of 12 simultaneous homogeneous algebraic equations. Because there are no such general solutions presented in the literature, verification is done by some special cases such as delaminated composite beams (without core) and perfect composite sandwich beams (without delamination). Based on this general solution, the effects of face/core, core, and delamination on the free vibration behavior of composite sandwiches are studied thoroughly.

I. Introduction

There are many advantages of composite sandwich structures over the conventional structural materials, such as high bend- ing stiffness, low specific weight, and good thermal and acous- tical insulation. However, these new materials also induce some new problems. One of them is delamination, which may occur either on the interface of composite laminated faces or on the interface between face and core. It is, therefore, important to know the effect of delamination on some mechanical properties like free vibration.

To study the delamination effect on free vibration, Kolling and Fredrich studied a circular cylindrical shell with a circumferen- tially symmetric delamination of small length at the middle sur- face and proposed that the natural frequency was a parameter that reflects the debonding or weakening of the composite. Ramakrishna et al. studied the free vibration of composite beams with through- thickness delamination and compared the theoretical results with the results of vibration experiments on a debonded laminated casteller beam. The analytical prediction found to be consistently much lower than the experimental values indicates that the residual bending stiffness was grossly underestimated in these experiments. Cawley and Adams considered the shift in natural frequency due to the delamination provides the basis for nondestructive testing via vibration technique by considering the coupling of longitudinal and flexu- ral modes in the sandwich beam. In this paper, a one-dimen- sional model to analyze the free vibration of a thick isotropic beam. Later, Meijnderaar and Suyama studied the effect of delamina- tion on flexural vibration of an isotropic beam under the as- sumption that the split regions of the beam should be constrained to move together in the transverse direction. Tracy and Puckett studied the effect of delamination on natural frequency of sym- metric laminated beam containing bi-phase delamination by using the Euler beam theory and verified the results by experiments and the finite element method. Recently, Shav and Grady used the Gulerian method to analyze the delamination effect on natural fre- quency and vibration mode shape of composite laminated beam and verified the results by experiments, in which the groupings between longitudinal vibration and transverse bending motion were also considered.

II. Vibration Analysis

Recently, a onedimensional mathematical model was developed by Hsiao and Hsiao for the buckling and postbuckling of delami- nated composite sandwich beams. In that model, the delaminated composite sandwich beam is separated into four regions as shown in Fig. 1. Regions 1 and 4 are considered to be composite sandwich beams with the faces resisting in-plane forces Nx and bending moment Mz, and the core undergoing the transverse shear force Qz, whereas regions 2 and 3 are considered to be special cases of sandwiches that also carry in-plane loads, bending moment and transverse shear force. In all regions, the deformation of the core and faces are assumed to have the form of the Timoshenko beam, i.e.,

\[ u = w + \frac{3}{2} \frac{w'}{sx} \]

(1)

where w and w' denote the displacements in the x and z directions, respectively; \( \nu_{s} \) is the midplane axial displacement; and \( \nu_{s} \) is the transverse shear strain. Although w is a linear function of x, w' is independent of z and w satisfies the conditions of a simply supported beam. With this assumption for the sandwich beam deformation, the equations of
motion can be derived by a way similar to that described in the paper by Huo and Hu.\textsuperscript{11} The results are

\[
\frac{\partial N_x}{\partial x} = 0
\]

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = q = \frac{\partial w}{\partial t}
\]

\[
\frac{\partial \sigma_y}{\partial y} = Q_y + \frac{\partial w}{\partial t}
\]

(2a)

where

\[
I = \frac{\rho h^4}{12}, \quad \mu = \frac{w}{\nu} - \frac{2w}{\nu}
\]

and \( \mu, h, \) and \( \beta \) are, respectively, the moment of inertia (with respect to the midplane), mass density, thickness, and curvature angle of the sandwich beam. \( q \) is the normal pressure distributed over the top or bottom surface of the sandwich beam. The resultant forces \( N_x, M_x, \) and \( Q_y \) are related to the midplane axial strain \( c_{11}, \) curvature \( \kappa, \) and the transverse shear strain \( \eta_x \) by

\[
N_x = A_{11} c_{11} + B_{11} \kappa_x + D_{11} \eta_x
\]

\[
M_x = B_{11} c_{11} + D_{11} \kappa_x
\]

\[
Q_y = S_y \kappa
\]

(2c)

and

\[
\kappa_x = \frac{\partial \kappa}{\partial x} = \frac{\partial \kappa}{\partial x} = \frac{\partial \kappa}{\partial x} = \frac{\partial \kappa}{\partial x}
\]

(2b)

\[
\kappa_x = \frac{\partial \kappa}{\partial x} = \frac{\partial \kappa}{\partial x} = \frac{\partial \kappa}{\partial x} = \frac{\partial \kappa}{\partial x}
\]

(2b)

where \( A_{11}, B_{11}, \) and \( D_{11}, \) and \( S_y \) are, respectively, the longitudinal, transverse, bending, and transverse shear stiffness of the composite sandwich beam. The stiffnesses \( A_{11}, B_{11}, \) and \( D_{11}, \) are contributed by the faces of the sandwich, whereas the shear stiffness \( S_y \) is mostly contributed by the core. The formulas for calculating \( A_{11}, B_{11}, \) and \( D_{11}, \) are the same as those given in the classical laminated theory except that the plane \( z = 0 \) is located in the midplane of the entire sandwich and not the core.\textsuperscript{12} To calculate the transverse shear strain \( \eta_x \), the shear stress distribution is assumed to be uniform across the core and parabolic across the face.\textsuperscript{13} Hence,

\[
S = c_{11} \eta_x + \sum_{i=1}^{m} \left( \frac{\partial \eta_x}{\partial x} \right) z_i \frac{1}{2} \left( 1 - z_i^2 \right) R_i^2
\]

(5a)

where \( R_i = c_{11} \cos \theta + \beta_i \sin \theta, \) where \( c_{11}, \cos \theta, \) and \( \theta \) are, respectively, the transverse shear modulus and fiber direction of the laminate. The terms \( c_{11} \) and \( \beta_i, \) are the thickness and effective transverse shear modulus of the core. \( N \) is the number of layers of composite sandwich (excluding the core), and \( z_i \) is the coordinate of \( i \)th layer as shown in Fig. 1. If only the composite transverse faces are considered, the first term \( c_{11} \) should be deleted since it is contributed by the core. It should also be noted that in Eq. (5a) the longitudinal inertia term \( \frac{\partial \kappa_x}{\partial x} \) has been neglected since it is small, for the lower flexural modes of beams.\textsuperscript{1}

The first equation of Eq. (2a) shows that \( N_x \) is a constant throughout the beam and it is also the compressive axial load \( P, \) i.e., \( N_x = -P. \) Knowing that \( P \) is a constant, by the first and second equations of Eq. (2c) the transverse axial strain \( \sigma_y \) and the bending moment \( M_x \) can be expressed in terms of the curvature \( \kappa, \) which are

\[
\sigma_y = -A_P - B_{11} \kappa_x
\]

\[
M_x = B_{11} \kappa_x + D_{11} \eta_x
\]

(3a)

(3b)

where

\[
A = \frac{A_{11}}{A_{10}}, \quad B = \frac{B_{11}}{B_{10}}, \quad D = \frac{D_{11}}{D_{10}}
\]

(4a)

If the axial load \( P \) is treated as a known value, the three equations of motion shown in Eq. (2a) may be reduced to only one equation expressed by the transverse displacement \( w. \) Use of the second equation of Eq. (2a) and the third equation of Eq. (2c) may provide the relation between \( \sigma_y \) and \( w. \) By this relation and the second equation of Eq. (2b) the curvature \( \kappa_x, \) can be expressed in terms of \( w. \) They are

\[
\frac{\partial^2 w}{\partial x^2} = \frac{P w \sigma_y}{2 A_P} + \frac{P \kappa_x \sigma_y}{B_{11}} - \frac{\sigma_y}{S_y}
\]

\[
\kappa = \frac{\partial^2 w}{\partial x^2} - \frac{P w \sigma_y}{2 A_P} - \frac{P \kappa_x \sigma_y}{B_{11}} + \frac{\sigma_y}{S_y}
\]

(4b)

Substituting the third equation of Eq. (3c), and the second and third equations of Eqs. (3a) and (3b) into the third equation of Eq. (2c) and differentiating both sides of the equation with respect to \( x, \) we can write the equation of motion as

\[
D \frac{\partial^4 w}{\partial x^4} = S_y \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}
\]

(5a)

With the relations provided in Eq. (4), the equation of motion for the composite sandwich beams can now be expressed by only one parameter \( w. \)

\[
D \frac{\partial^4 w}{\partial x^4} = \frac{P w \sigma_y}{2 A_P} + \frac{P \kappa_x \sigma_y}{B_{11}} + \frac{\sigma_y}{S_y}
\]

(5b)

If we now consider the problems of free vibration, \( \sigma_y = 0 \) for regions 1 and 4, the equations of motion (5b) can be reduced to

\[
D \frac{\partial^4 w}{\partial x^4} = \frac{P \rho k D_i}{S_i} \frac{\partial^4 w}{\partial x^4} + \frac{\rho k D_i}{S_i} \frac{\partial^4 w}{\partial x^4} + \frac{\rho k D_i}{S_i} \frac{\partial^4 w}{\partial x^4} + \frac{\rho k D_i}{S_i} \frac{\partial^4 w}{\partial x^4}
\]

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where the normal pressure \( q \) can be represented in harmonic form as
\[
q = q_0 \sin \omega t, \quad q_0 = \text{const}
\]  
(7b)

With the preceding assumptions, Eq. (5b) can be simplified. Moreover, by the fact that the rotary inertia \( I \) is higher order in \( h \), which is small even for the sandwich beam, and the in-plane forces are usually far smaller than the transverse shear forces for the flexural vibration problems, the terms \( \omega^2 / S \) and \( S / \omega^2 \) may be far less than unity and may be neglected for the convenience of mathematical manipulation.

For the sake of conciseness, we will check this assumption after the natural frequencies and the mode shapes are obtained. Now, the equations of motion for regions 2 and 3 can be simplified as
\[
D_2 \frac{d^2 w_2}{dt^2} = \left( I_1 + \rho h_2 D_2 \frac{d^2 w_2}{dt^2} + \left( \rho h_3 D_3 \frac{d^2 w_3}{dt^2} \right) \right) \frac{d^2 w_3}{dt^2} + \frac{\rho h_2 h_3}{S_2} \frac{d^2 w_3}{dt^2} \frac{d^2 w_2}{dt^2} = q
\]
\[
- \rho h_2 \frac{d^2 w_2}{dt^2} = \rho h_3 \frac{d^2 w_3}{dt^2} = \rho h_2 \frac{d^2 w_2}{dt^2} = \rho h_3 \frac{d^2 w_3}{dt^2} = q
\]
(8)

in which the local coordinates \( x_2 \) and \( x_3 \) are chosen to be the same as the global coordinate \( x \) (see Fig. 1). By adding the preceding two equations, the unknown normal contact pressure \( q \) and the axial load \( P \) can be eliminated. The result is
\[
(D_1 + D_2) \frac{d^2 w_1}{dt^2} = \left( I_1 + \rho h_2 D_2 \frac{d^2 w_2}{dt^2} + \left( \rho h_3 D_3 \frac{d^2 w_3}{dt^2} \right) \right) \frac{d^2 w_3}{dt^2} + \frac{\rho h_2 h_3}{S_1} \frac{d^2 w_3}{dt^2} \frac{d^2 w_2}{dt^2} = 0
\]  
(9)

For harmonic motion,
\[
w_1(x_1, t) = W_1(x_1) \sin \omega t, \quad i = 1, 2, 4
\]

in which \( \omega \) is the natural frequency and \( W_1(\cdot) \) is the mode shape for the \( i \)-th region. Substituting Eq. (10) into Eqs. (6) and (9), one can obtain the general solutions of the differential equations (6) and (9) as
\[
W_1(x_1) = \Gamma_1 \cosh \mu_1 x_1 + \Gamma_3 \cosh \mu_3 x_1 + \Gamma_2 \cos \mu_2 x_1 + \Gamma_4 \sin \mu_4 x_1
\]
where
\[
(\mu_2, \mu_3, \mu_4, \mu_5) = \left\{ \frac{\sqrt{2D_1 + D_2}}{2}, \frac{\sqrt{\left( I_1 + I_2 \right) + \sqrt{4I_2 + 4I_3 + 4I_4}}}{2}, \frac{\sqrt{\left( I_1 + I_2 \right) + \sqrt{4I_2 + 4I_3 + 4I_4}}}{2}, \frac{\sqrt{2D_1 + D_2}}{2} \right\}
\]  
(10)

where \( \mu_1 = \frac{1}{2} \sqrt{2D_1 + D_2} \), \( \mu_2 = \frac{1}{2} \sqrt{I_1 + \sqrt{I_2^2 + 4I_3 + 4I_4}} \), \( \mu_3 = \frac{1}{2} \sqrt{I_1 + \sqrt{I_2^2 + 4I_3 + 4I_4}} \), and \( \mu_4 = \frac{1}{2} \sqrt{2D_1 + D_2} \).

Thus, it can be seen that the natural frequencies \( \omega_i \) are related to \( \omega_n \) by the third equation of Eq. (2c). The bending moment \( M_2 \) and the transverse shear strain \( \gamma_{21} \) are related to the transverse displacement \( w_2 \) by the second equation of Eq. (5a) and Eq. (4). With these relations, it is noted that the continuity conditions shown in Eq. (14) can all be expressed in terms of the transverse displacements \( w_2 \) except that the bending and shear force continuity have the extra unknown axial load \( P \) and normal pressure \( q \) in which the axial load \( P \) may be found by the solution of compatibility of axial displacements at the tip of delamination. That is,
\[
\left( \begin{array}{c}
\omega_n \\
\frac{1}{2} \left( \omega + \omega_n \right)
\end{array} \right) \left( \begin{array}{c}
\frac{2}{2 + \phi} \frac{2}{2 + \phi} \frac{2}{2 + \phi} \\
\frac{2}{2 + \phi} \frac{2}{2 + \phi} \frac{2}{2 + \phi}
\end{array} \right) \left( \begin{array}{c}
u (x_2) \n(\omega) + \frac{\phi}{2}\frac{2}{2 + \phi} \frac{2}{2 + \phi} \\
\n(\omega_n) + \frac{\phi}{2}\frac{2}{2 + \phi} \frac{2}{2 + \phi}
\end{array} \right) = \left( \begin{array}{c}n(x_2) \n(\omega) + \frac{\phi}{2}\frac{2}{2 + \phi} \frac{2}{2 + \phi} \\
\n(\omega_n) + \frac{\phi}{2}\frac{2}{2 + \phi} \frac{2}{2 + \phi}
\end{array} \right)
\]

The modeplane axial displacement \( w_2 \) of regions 2 and 3 can also be expressed in terms of \( w_2 \) by substituting the first equation of Eq. (5a)
into the first equation of Eq. (3a) and integrating with respect to \( x \). The results are

\[
\psi(x) = -A_0 x + B_0 + \frac{2\psi_0}{\alpha_2} \int_0^x \frac{\alpha_1^2}{\alpha_2} \, dx + \text{const}
\]

(16)

Substituting Eq. (16) into Eq. (15) and subtracting the second equation (5) by the first equation (15), we obtain

\[
P = \frac{(B_1 - B_0 - q/2)}{2\alpha_2 (A_0 - A_1)} \frac{\alpha_1^2}{\alpha_2} \int_0^x \frac{\alpha_1^2}{\alpha_2} \, dx \mid_{x = a} - \frac{\alpha_1^2}{\alpha_2} \mid_{x = -a}
\]

(17)

By using the relation given in Eq. (17), the other unknown \( q \) may be found by the satisfaction of compatibility of transverse shear strain at the crack tip of delamination. However, this approach may lead to a nonlinear equation of \( T_{12} \), which may cause trouble in mathematical manipulation. To avoid this, we ignore \( q/\alpha_2 \) since they may be very small as the other terms shown in the second equation of Eq. (13). This assumption will also be checked after the natural frequencies and mode shapes are obtained.

The compatibility conditions (14) will now provide 8 linear homogeneous algebraic equations in 12 unknown coefficients \( T_{ij} \) (i = 1, 2; j = 1, 2, 3, 4). The remaining four equations come from the boundary conditions for both ends of the sandwich beam. In the following, the three different boundary conditions will be studied. They are the following:

1) Simply supported ends:

\[
\psi(0) = 0, \quad (M_1)_{x=0} = 0, \quad \psi_0 = 0, \quad (M_2)_{x=0} = 0
\]

(18a)

2) Clamped-clamped ends:

\[
\psi(0) = 0, \quad \frac{\partial \psi_0}{\partial x} = 0, \quad (M_1)_{x=0} = 0, \quad (M_2)_{x=0} = 0
\]

(18b)

3) Clamped-free ends:

\[
\psi(0) = 0, \quad \frac{\partial \psi_0}{\partial x} = 0, \quad (M_2)_{x=0} = 0, \quad \psi_0 = 0, \quad (M_2)_{x=0} = 0
\]

(18c)

In the preceding text, all of the boundary conditions provide four equations. Combining them 4 equations with the 8 constancy equations, one obtains 12 linear homogeneous algebraic equations in 12 unknown coefficients \( T_{ij} \). The frequencies and mode shapes can be obtained as the eigenvalues and eigenvectors of this equation set.

IV. Special Cases

Since no published analytical results have been found in the literature for the vibration analysis of delaminated composite sandwich beams, the verification will be done by some degenerate cases such as perfect sandwich beams (without delamination) and saturated composite beams (without core).

A. Perfect Sandwich Beams (Without Delamination)

For the special case that the delamination does not exist in the sandwich beam, the problem becomes much simpler than those discussed previously. In this case, there is no need to separate the beam into four regions. All we need to do is to substitute the boundary conditions (18) into the general solutions of the first equation of Eq. (11a) for the perfect sandwich beams. The natural frequencies \( \omega \) and the mode shapes \( \psi(x) \) can be obtained as follows:

1) Simply supported ends:

\[
\frac{\rho h}{D} \int_0^L \left( \frac{1}{\alpha_2} + \frac{1}{\alpha_3} \right) \frac{\delta L}{\sinh L} + \frac{\rho h}{D} \frac{\delta L}{\sinh L} \delta L = \frac{\omega^2}{\sinh L} \sinh L \sinh L \sinh L \sinh L = 0
\]

(19a)

B. Delaminated Composites (With Core)

An example of delaminated composite beam [90/0%/0%/90°], made of T300/934 graphite/epoxy with debonding along interface 2 (given in the paper by Shih and Grahby) is analyzed for the purpose of verification. The results presented in Fig. 1 show that the fundamental resonant frequencies obtained by the present model match very well with experimental results as compared with those obtained by the models given by Shih and Grahby. These results give us confidence in the present model, which is general and can be used to reduce the free vibration problem of delaminated composite sandwiches.

V. Results and Discussion

The geometric notations of delaminated composite sandwich is shown in Fig. 1, in which the face and core of the sandwich are made of carbon-epoxy and aluminium honeycomb, respectively. The reference material properties of carbon-epoxy are
Table 1: Comparisons of $P_i/S_i$, $2U_i/S_i$, $T_{\text{me}}/d_{\text{core}}/d_{\text{ax}}$, and $(2a_i/S_i)/T_{\text{me}}$ with unity ($k_i/S_i$/$k_i$/honeycomb).

<table>
<thead>
<tr>
<th>$2a_i/L$</th>
<th>Boundary conditions</th>
<th>$P_i/S_i$</th>
<th>$P_i/S_i$</th>
<th>$(2U_i/S_i)/(d_{\text{me}}/d_{\text{core}})/d_{\text{ax}}$</th>
<th>$(2a_i/S_i)/T_{\text{me}}$</th>
<th>$k_i/S_i$/$k_i$/honeycomb</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>i.e.</td>
<td>5.09 $\times 10^{-4}$</td>
<td>8.75 $\times 10^{-5}$</td>
<td>4.69 $\times 10^{-5}$</td>
<td>5.78 $\times 10^{-7}$</td>
<td>3.51 $\times 10^{-4}$</td>
</tr>
<tr>
<td>0.2</td>
<td>i.e.</td>
<td>7.20 $\times 10^{-4}$</td>
<td>5.06 $\times 10^{-4}$</td>
<td>9.86 $\times 10^{-4}$</td>
<td>8.42 $\times 10^{-4}$</td>
<td>1.12 $\times 10^{-3}$</td>
</tr>
<tr>
<td>0.3</td>
<td>i.e.</td>
<td>1.00 $\times 10^{-3}$</td>
<td>5.32 $\times 10^{-4}$</td>
<td>3.75 $\times 10^{-3}$</td>
<td>1.19 $\times 10^{-4}$</td>
<td>3.87 $\times 10^{-4}$</td>
</tr>
<tr>
<td>0.4</td>
<td>i.e.</td>
<td>1.64 $\times 10^{-3}$</td>
<td>5.99 $\times 10^{-3}$</td>
<td>2.49 $\times 10^{-3}$</td>
<td>1.80 $\times 10^{-4}$</td>
<td>1.20 $\times 10^{-3}$</td>
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Fig. 3: Comparison of deformation length effect on natural frequency.

Fig. 4: Normal pressure distribution along deformation region ($k_i/S_i$/$k_i$/honeycomb).

Fig. 5: Effect of deformation length effect on first mode natural frequency of composite sandwich beam ($k_i/S_i$/$k_i$/honeycomb), carbon/epoxy laminated faces).

A. Deformation Effect

The deformation lies symmetrically with respect to the midpoint of the beam. Figure 5 shows the relation between the first mode natural frequency $\omega_{\text{me}}/\omega_{\text{me}}$ and deformation length $2a_i/L$ for clamped-free ends as the core thickness $c$ is reduced to be zero gradually. From this figure, it is shown that the existence of deformation will lower the natural frequencies. In the lower bound deformation, the first mode natural frequency of the composite sandwich beam decreases gradually when the core thickness of 0.37 mm decreases and will approach that of a deformed composite laminate as the core thickness is reduced to zero. The results of deformed composite
beams are represented by setting \( c = 0 \), which have been checked in the last section through the minimization of Fig. 3. To see more clearly, a counterpart of Fig. 5 with the same composite laminate used by Shen and Gere is plotted in Fig. 6. These results also mean that the model of delaminated composite sandwich proposed here is reliable.

The effects of delamination along spanwise locations on natural frequency for cantilever beam are shown in Fig. 7. It shows that the natural frequency of a delaminated composite sandwich will also approach that of a delaminated composite laminate when the core thickness is reduced to zero. The weakening effect of the delamination on the first mode natural frequency appears to become a minimum when the delamination is placed near the free end of a cantilever beam; that is, it occurs on the high curvature of the first mode shape. As to the case of simply supported ends, the minimum weakening effect of the delamination occurs when the delamination is located symmetrically with respect to the midpoint of the beam,\(^4\) where

\[ f = 0.02 \text{ in} \quad L = 3 \text{ in} \]

Nuclear phenomena occur for delaminated composite beams, which have been explained by Trace and Pridmore,\(^4\) that the effect of the delamination is reduced as the delamination moves from regions of high stress to regions of low curvature. The same conclusions are also supported by Bhargava and Sivaramakrishnan\(^5\) for delaminated isotropic beams.

As for free vibration mode shapes, Fig. 8 shows that the mode shapes of a delaminated composite sandwich beam will approach those of a delaminated composite laminate as the core thickness is reduced to zero. It also shows that the mode shapes of delaminated sandwich beams differ from those of perfect sandwich beams, especially in the region of delamination because the effective bending stiffness \( D_0 \), \( t = 2, 3 \), of the delaminated region is smaller than that of region 1. Therefore, the deflection pattern is significantly influenced by the delamination.

B. Coup Effect

The effect of core on the natural frequency is usually discussed by considering the thickness and effective transverse shear modulus of the core. Figure 9 shows the relation between natural frequency and transverse shear modulus \( G_{0r} \) for various central delamination sizes for clamped-free boundary conditions, where \( G_{0r} = 6.300 \text{ GPa} \), and the core thickness \( c_0 \) is kept at 0.75 mm. The figure shows that the natural frequency depends significantly on the transverse shear modulus when the delamination length is short, whereas there is almost invariance on frequency for longer delaminations. This phenomenon may be explained by thinking that the contribution of the core is only minor in the delamination region. Therefore, the longer the delamination, the smaller the transverse shear modulus effect.

To study the effect of core thickness on the natural frequency, the transverse shear modulus \( G_{0r} \) is kept fixed and set to be equal to
Table 2: Effect of stacking sequence on the natural frequency of delaminated sandwich (composite laminate/honeycomb) beams with clamped-free ends.

<table>
<thead>
<tr>
<th>$\theta$/</th>
<th>$x_1^*$</th>
<th>$x_2^*$</th>
<th>$x_3^*$</th>
<th>$x_4^*$</th>
<th>$x_5^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.571</td>
<td>0.492</td>
<td>0.459</td>
<td>0.457</td>
<td>0.409</td>
</tr>
<tr>
<td>0.3</td>
<td>0.348</td>
<td>0.292</td>
<td>0.269</td>
<td>0.269</td>
<td>0.237</td>
</tr>
<tr>
<td>0.4</td>
<td>0.236</td>
<td>0.196</td>
<td>0.180</td>
<td>0.188</td>
<td>0.158</td>
</tr>
<tr>
<td>0.5</td>
<td>0.173</td>
<td>0.143</td>
<td>0.131</td>
<td>0.171</td>
<td>0.115</td>
</tr>
</tbody>
</table>

Notes: $x_1^* = [0/-45/45/0]/[0/-45/45/0]/[45/-45/0]/[45/-45/0]$ and $x_5^* = [0/0/45/45]$.

Fig. 11: Effect of stacking sequence on the natural frequency of delaminated composite sandwich beams.

The highest one, whereas the rest of $x_5$ is always the lowest one. The same conclusions are also found for simply supported and clamped-clamped boundary conditions. The reasons are the same as those described in Refs. 10 and 11 of the paper for the effect of stacking sequence on buckling load, i.e., the natural frequency depends on the effective bending stiffnesses $D_2$ and $D_3$ of the delaminated regions, not $D_1$. This means that the order of the relative natural frequencies shown in this table is $x_1^* > x_2^* > x_3^* > x_4^* > x_5^*$, which is consistent with the order of $D_1$ of the previous paper, not $D_1$.

VI. Conclusion

A one-dimensional model of free vibration behavior of delaminated composite sandwich beams considering transverse shear effect and rotary inertia has been established in this paper. The solutions are general and can be reduced to solve the free vibration problem of delaminated composite beams and perfect sandwich beams. When the core thickness is close to zero, the solutions of the present model approach those of delaminated composite beam, which verifies the present model. Meanwhile, the effect of core, facts, and delamination on natural frequencies and the associate mode shapes are also discussed in this paper.

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References
