SEXTIC FORMALISM IN ANISOTROPIC ELASTICITY FOR ALMOST NON-SEMISIMPLE MATRIX N

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Abstract—The sextic formual of Stroh for anisotropic elasticity leads to the eigenversion N = \mathbf{p}^2 in which \( N \) is a 6 \times 6 real matrix. The orthogonality and closure relations as well as many other relations involving the eigenvalues \( p \) and the eigenvectors \( \mathbf{z} \) are based on the assumption that \( N \) is simple or semisimple so that the six eigenvalues \( p_i \) span a six-dimensional space. Problems arise when \( N \) is non-semisimple. In fact there are problems even when \( N \) is almost non-semisimple. We present a modified formalism which is valid regardless of whether \( N \) is simple, almost non-semisimple or non-semisimple. The modified formalism does not apply when \( N \) is semisimple.

1. INTRODUCTION

The sextic formalism for anisotropic elasticity originally due to Stroh[1, 2] assumes that the 6 \times 6 real matrix \( N \) is simple. This means that the eigenvalues \( p_i \) (\( i = 1, 2, \ldots, 6 \)) of \( N \) are distinct so that there are six independent eigenvectors \( \mathbf{z}_i \). The formalism applies also to semisimple \( N \) in which there is a repeated eigenvalue, say \( p_1 = p_2 \), but there exist two independent eigenvectors \( \mathbf{z}_1 \) and \( \mathbf{z}_2 \). When \( N \) is non-semisimple, i.e. when \( p_1 = p_2 \) and there exists only one independent eigenvector associated with \( p_1 \) and \( p_2 \), the Stroh formalism does not apply. Anisotropic elastic materials which lead to a non-semisimple \( N \) are called degenerate materials. Isotropic materials are a special group of degenerate materials. Nishioka and Lothe[3, 4] studied the limiting behavior of the Stroh formalism when the material becomes isotropic. Lothe and Bazaerti[5] and Chadwick and Smith[6] introduce the generalized eigenvectors and obtain an important result that some relations for simple \( N \) continue to hold for non-semisimple \( N \) if the eigenvectors are replaced by the generalized eigenvectors. However, as we will see in this paper, not all relations for simple \( N \) can be converted to relations for non-semisimple \( N \) by simply replacing the eigenvectors by the generalized eigenvectors. Examples will be given in this paper. The main purpose of this paper however is to look at the situation in which \( N \) is almost non-semisimple.

When \( N \) is simple or semisimple, the Stroh formalism applies. When \( N \) is non-semisimple, the generalized eigenvectors take the place of eigenvectors. The transition of the formalism from a simple or semisimple to non-semisimple \( N \) is not continuous. This suggests that some difficulties may arise when \( N \) is almost non-semisimple. Indeed, as we will see in Section 2 where we summarize the Stroh formalism, when \( N \) is almost non-semisimple the magnitudes of the orthonormalized eigenvectors associated with the almost equal eigenvalues are very large and becomes infinite as the two eigenvalues become equal. To overcome this difficulty we present in Section 3 a modified sextic formalism which applies to almost non-semisimple \( N \). The formalism remains valid when \( N \) is non-semisimple. In fact the assumption of almost non-semisimple is not required in the derivation and hence the formalism applies to simple \( N \) as well. The modified formalism however does not apply to \( N \) which is semisimple.

In Section 4 we show the conversion from the Stroh formalism to the present modified formalism. With the conversion many relations which are valid for simple or semisimple \( N \) can be rewritten for non-semisimple or almost non-semisimple \( N \). Applications to some rules are given in Section 5. Finally we show in Section 6 how one can split the generalized 6-vectors \( \mathbf{z} \) for almost non-semisimple \( N \) into two 3-vectors \( \mathbf{a} \) and \( \mathbf{b} \) and determine them separately.
In a fixed rectangular coordinate system \((x_1, x_2, x_3)\) let the stress \(\sigma_{ij}\) and strain \(e_{ij}\) of the material be related by

\[
\sigma_{ij} = C_{ijkl} e_{kl}
\]

\[
C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}
\]

where \(C_{ijkl}\) are the elasticity constants. Unless stated otherwise repeated indices imply summation. For two-dimensional deformations in which the displacements \(u_k\) \((k = 1, 2, 3)\), depend on \(x_1\) and \(x_2\) only, a general solution for \(u_k\) can be written in matrix notation as

\[
\mathbf{u} = \mathbf{g}(z)
\]

\[
z = x_1 + \rho x_2
\]

in which \(\mathbf{g}\) is an arbitrary function of \(z\). The eigenvalue \(\rho\) and the eigenvector \(\mathbf{a}\) are determined from [7]

\[
\mathbf{D}(\rho)\mathbf{a} = \mathbf{0}
\]

\[
\mathbf{D}(\rho) = \mathbf{Q} + \rho(\mathbf{R} + \mathbf{R}^T) + \rho^2 \mathbf{T}
\]

where superscript \(T\) stands for the transpose and the \(3 \times 3\) matrices \(\mathbf{Q}, \mathbf{R}\) and \(\mathbf{T}\) are given by

\[
\mathbf{Q}_{ij} = C_{ijkl} \mathbf{R}_{kl} = C_{ijkl}, \quad \mathbf{T}_{ij} = C_{ijkl}.
\]

Matrices \(\mathbf{Q}\) and \(\mathbf{T}\) are symmetric and positive definite if the strain energy is positive. Introducing the new vector

\[
\mathbf{b} = (\mathbf{R}^T + \rho \mathbf{T})\mathbf{a} = -\frac{1}{\rho}(\mathbf{Q} + \rho \mathbf{R})\mathbf{a}
\]

in which the second equality comes from eqn (5), the stresses are obtained from the stress function \(\phi\) by [1, 2]

\[
\sigma_{ij} = -\partial \phi / \partial x_i, \quad \phi_{ij} = \partial \phi / \partial x_j
\]

\[
\phi = \mathbf{b}(z).
\]

Equations (8), and (9), can be written in the standard eigen-relation as

\[
\mathbf{N} \xi = \rho_s \xi
\]

\[
\mathbf{N} = \begin{bmatrix} N_1 & N_1 \\ N_2 & N_2 \end{bmatrix}, \quad \xi = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}
\]

\[
N_1 = \mathbf{R}^T, \quad N_2 = \mathbf{T}^{-1} - N_1
\]

\[
N_s = \mathbf{R}^T \mathbf{R}^T - \mathbf{Q} = \mathbf{N}_s
\]

Thus \(\xi\) is the right eigenvector of the \(6 \times 6\) real matrix \(\mathbf{N}\). The left eigenvector \(\mathbf{a}\) satisfies
Introducing the $6 \times 6$ matrix $J$ by

$$ J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} $$  \hspace{1cm} (15) $$

where $I$ is the identity matrix, it can be shown that

$$ JN = (JN)^T = N^TJ. $$  \hspace{1cm} (16) $$

From eqns (11), (14) and (16) we may set without loss of generality

$$ \eta = J\xi = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}. $$  \hspace{1cm} (17) $$

Since $\rho$ cannot be real if the strain energy is positive[7], we have three pairs of complex conjugates for $\rho$ as well as for $\xi$ and $\eta$. If $\rho_{x}$, $\xi_{x}$ and $\eta_{x}$ ($x = 1, \ldots, 6$) are the eigenvalues and the eigenvectors, we let

$$ \rho_{x+3} = \rho_{x}, \hspace{1cm} \text{Im} \rho_{x} > 0 $$
$$ \xi_{x+3} = \xi_{x}, \hspace{1cm} \eta_{x+3} = \eta_{x} $$ \hspace{1cm} (18) $$

where Im denotes the imaginary part and an overbar stands for the complex conjugate. When $N$ is simple or semisimple, $\xi_{x}$ span a six-dimensional space and are orthogonal to $\eta_{x}$. Since $\xi_{x}$ obtained from eqn (11) are unique up to a multiplicative constant, we may normalize $\xi_{x}$ such that (with $\eta_{x}$ determined from eqn (17))

$$ \eta_{x} \xi_{x} = \delta_{x\alpha} $$  \hspace{1cm} (19) $$

where $\delta_{x\alpha}$ is the Kronecker delta. The orthonormal relations can be written in matrix notation as

$$ V^T U = I $$  \hspace{1cm} (20) $$

in which the $6 \times 6$ matrices $U$ and $V$ are

$$ U = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 \end{bmatrix} $$
$$ V = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 \end{bmatrix} $$  \hspace{1cm} (21) $$

If we introduce the $3 \times 3$ matrices

$$ A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \hspace{1cm} B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} $$  \hspace{1cm} (22) $$

we may write $U$ and $V$ as, using eqns (12) and (17)

$$ U = \begin{bmatrix} A & \tilde{A} \\ B & \tilde{B} \end{bmatrix}, \hspace{1cm} V = JU. $$  \hspace{1cm} (23) $$

Equation (20) implies that $V^T$ and $U$ are the inverse of each other and hence the order of the product can be interchanged. We have
or, carrying out the matrix multiplications using eqns (15) and (23)

\[
\begin{align*}
AA' + \tilde{A}A' &= 0 = BB' + \tilde{B}B' \\
BA' + \tilde{B}A' &= I = AB' + \tilde{A}B'.
\end{align*}
\]

(25)

These are the closure relations. Equations (25) tell us that there exist real matrices \( H, L \) and \( S \) such that

\[
\begin{align*}
H &= 2AA' = H^T \\
L &= -2BB' = L^T \\
S &= i(2AB' - D).
\end{align*}
\]

(26)

We see that \( H \) and \( L \) are symmetric, and can be shown to be positive definite if the strain energy is positive [6]. The three real matrices \( H, L \) and \( S \) play important roles in the problems of anisotropic elasticity and surface waves (see, e.g., Refs [6, 8–11]).

The above formalism from eqns (19) to (26) are valid if \( N \) is simple or semisimple because we have six independent eigenvectors \( \xi \). If \( N \) is non-semisimple, say we have \( p_1 = p_2 \) and also \( \xi_1 = \xi_2 \), we do not have six independent eigenvectors to span the six-dimensional space. Consequently, eqns (19)–(26) are not valid. Isotropic materials are the well-known example of having a non-semisimple \( N \) for which \( p_1 = p_2 = i \) and \( \xi_1 = \xi_2 \). In fact \( p_1 = i \) also but \( \xi_1 \) is independent of \( \xi_2 \).

One encounters difficulties not only when \( N \) is non-semisimple but also when \( N \) is almost non-semisimple. This means that \( p_1 \) and \( p_2 \) are almost equal as are \( \xi_1 \) and \( \xi_2 \). To see the problems which may arise when \( N \) is almost non-semisimple, let \( \xi_1 \) and \( \xi_2 \) be unit vectors satisfying eqn (11) for \( p = p_1 \) and \( p_2 \) respectively. Assuming that \( p_1, p_2 \) are almost equal as are \( \xi_1, \xi_2 \), we let

\[
\xi = \xi_1 + s(\delta y), \quad \delta = p_2 - p_1;
\]

(27)

in which \( y \) is a unit vector and \( s \) is a function of \( \delta \) such that as \( \delta \) approaches zero so does \( s \). To have an orthonormal system we set

\[
\begin{align*}
\xi_1 &= k_1 \xi_1, \quad \xi_2 = k_2 \xi_1 + ey_1, \\
\eta_1 &= J \xi_1, \quad \eta_2 = J \xi_2
\end{align*}
\]

(28)

where \( k_1, k_2 \) are complex constants to be determined. Application of eqn (19) for \( \alpha, \beta = 1, 2 \), leads to

\[
\begin{align*}
k_1 \xi_1 J \xi_1 &= 1 \\
k_2 \xi_2 J \xi_2 &= 1 \\
k_1 k_2 \xi_1 J \xi_2 &= 1 + s y_1 J \xi_1 + s y_2 J \eta_1 = 1.
\end{align*}
\]

(29)

Ignoring the \( s^2 \) term when \( \delta \) is small, we have

\[
k_2 = -k_1 = (s y_1 J \xi_1)^{-1}.
\]

(30)

Hence \( k_1 \) and \( k_2 \) are of order \( e^{-1/2} \). Consequently, the orthonormalized vectors \( \xi_1 \) and \( \xi_2 \) are very large vectors when \( \delta \) is small and become unbounded when \( \delta \) approaches zero. This creates problems for a numerical calculation of the eigenvectors when \( N \) is almost non-semisimple. Equations (30) also tell us that \( k_1 = \pm ik_2 \) and hence, as \( \delta \rightarrow 0 \), the orthonormalized eigenvectors \( \xi_1 \) and \( \xi_2 \) are not exactly equal but differ by a factor of \( \pm i \). The
statement that $\xi_1$ and $\xi_2$ are almost equal should therefore be interpreted as almost linearly dependent.

In the next section we present a modified formalism for the case when $N$ is almost non-semisimple. We will see in the derivation that the assumption of almost non-semisimple is unnecessary. The eigenvalues $p_1$ and $p_2$ need not be almost equal. The only requirement is that if $p_1$ and $p_2$ are almost equal, so are $\xi_1$ and $\xi_2$.

3. MODIFIED SEXTIC FORMALISM

We assume in this section that there is a possibility that $p_1$ and $p_2$ are either equal or almost equal. When that happens, we assume that $\xi_1$ and $\xi_2$ are also equal or almost equal. By eqns (31), $p_1$ and $p_2$ as well as $\xi_1$ and $\xi_2$ are equal or almost equal. It suffices to discuss the modifications required for $\xi_1$ and $\xi_2$ only.

From eqn (11) we have

$$\begin{align*}
N\xi_1' &= p_1\xi_1' \\
N\xi_2' &= p_2\xi_2'
\end{align*}$$

(31)

in which $\xi_1'$ and $\xi_2'$ are scalar multiples of $\xi_1$ and $\xi_2$ obtained in the last section. The scalar multiples are not unity or $\pm 1$ because of a different orthonormal system we are introducing here. Instead of eqn (31) we consider

$$\begin{align*}
N\xi_1' &= p_1\xi_1' \\
N\xi_2' &= p_2\xi_2' + \xi_2'
\end{align*}$$

(32)

where

$$\begin{align*}
\xi_2' &= (\xi_1' - \xi_2')(\delta) \\
\xi_1' &= \xi_1' + \delta\xi_2'
\end{align*}$$

(33)

$$\delta = p_2 - p_1.$$  

(34)

Equation (32) is obtained when we subtract eqn (31), from eqn (31), and divide the resulting equation by $(p_2 - p_1)$. Likewise, we will consider for the left eigenvectors the following equations:

$$\begin{align*}
N\eta_1 &= p_1\eta_1 + \eta_1' \\
N\eta_2 &= p_2\eta_2
\end{align*}$$

(35)

in which

$$\begin{align*}
\eta_1' &= (p_1 - p_2)(\delta) \\
\eta_1 &= \eta_2 - \delta\eta_1.
\end{align*}$$

(36)

Thus instead of $\xi_1$, $\xi_2$, $\xi_1'$, $\xi_2'$, we will use $\xi_1$, $\xi_2$, $\eta_1$, $\eta_1'$. They are determined from eqns (32) and (36). The vectors $\xi_1'$ and $\eta_1'$ are not employed, but their relations with $\xi_1'$, $\xi_2'$, $\eta_1'$, as given by eqns (33) and (36) will be useful in establishing certain identities. Hence $\delta$ can be arbitrary, zero or non-zero. Instead of solving eqn (35) for $\eta_1'$ and $\eta_2'$, they can be obtained from $\xi_1'$ and $\xi_2'$ by applying eqns (17) and (36). We have
The new vectors satisfy the following relations:
\[ \eta_1^T \xi_1 = -\eta_2^T \xi_2 = \eta_3^T \xi_3. \]  \hspace{1cm} (37)

Equations (38) and (39) are obtained when we pre-multiply eqns (32), and (32)_1, respectively, by \( \eta_1^T \) and use eqn (35). To form an orthonornomal system we must have
\[ \eta_1^T \xi_1 = 1, \quad \eta_2^T \xi_2 = 1, \quad \eta_3^T \xi_3 = 0. \]  \hspace{1cm} (39)

In view of eqn (38), we see that we do not have to consider all three equations in eqns (40). Since \( \xi_1, \xi_2, \eta_1, \eta_2 \) obtained from eqns (32) and (33) are not unique, we will show how one can obtain a set of vectors so that eqns (40) are satisfied.

Let \( \xi_1, \xi_2, \eta_1, \eta_2 \) satisfy eqns (32) and (35). They also satisfy eqns (38) and (39). It can be shown with the use of eqns (37) that
\[ \xi_1^T = k_1 \xi_1, \quad \xi_2^T = k_1 \xi_1 + k_2 \xi_2, \quad \eta_1^T = k_1 \eta_1 + k_2 \eta_2, \quad \eta_2^T = k_2 \eta_2. \]  \hspace{1cm} (41)

also satisfy eqns (32) and (35) in which \( k_1, k_2, k_3 \) are arbitrary complex constants which are related by
\[ k_2 = k_1 + \delta k_2. \]  \hspace{1cm} (42)

Imposition of eqns (40), (40)_1 and use of eqn (39) lead to
\[ k_1^2 = \eta_1^T \xi_1, \quad k_2^2 = \eta_2^T \xi_2. \]  \hspace{1cm} (43)

With eqns (43), eqn (38) can be written as
\[ k_1^2 - k_2^2 = \delta \eta_1^T \xi_2. \]  \hspace{1cm} (44)

If we solve for \( k_2 \) from eqn (44) and substitute it into eqn (42) we obtain
\[ k_2 = -k_1 (\eta_1^T \xi_2)/(k_1 + k_2). \]  \hspace{1cm} (45)

When \( \delta \neq 0 \), the orthogonal relation of \( \xi_1, \xi_2 \) \((\alpha = 1, 2)\) and eqns (35), and (39), assure us that \( \eta_1^T \xi_1 \) and \( \eta_2^T \xi_2 \) do not vanish. Hence \( k_1 \) and \( k_2 \) exist. In eqn (45) \( k_2 = -k_1 \) vanishes if \( k_2 = -k_2 \). However, \( k_2 \), obtained from eqn (43), is not unique in the sense that if \( k_2 \) is a solution so is \( -k_1 \). The same statement applies to \( k_1 \) and one can always choose the signs so that \( k_1 = k_2 \) instead of \( k_1 = -k_2 \). Hence \( k_2 \) also exists.

When \( \delta = 0 \), eqns (43) and (44) can be written as
\[ k_1^2 = k_2^2 - \eta_1^T \xi_1 = \eta_2^T \xi_2 \]  \hspace{1cm} (46)

The third equality in (46) comes from eqn (38). The existence of orthonormalized generalized
eigenvectors are assured by the theories on non-semisimple matrices [12]. Note that eqns (46) also apply to the case when \( \delta \neq 0 \) and \( k' \neq k_1 \).

With \( \xi'_1, \xi'_2, \xi'_3 \) properly orthonormalized, it can be shown that

\[
Y^{T}U' = I
\]  

(47)

in which

\[
U' = [\xi'_1, \xi'_2, \xi'_3, \xi'_4, \xi'_5, \xi'_6, \xi'_7] \quad V' = [\eta'_1, \eta'_2, \eta'_3, \eta'_4, \eta'_5, \eta'_6, \eta'_7]
\]  

(48)

In eqns (48) \( \xi'_i \) and \( \eta'_i \) are identical to the ones obtained in the last section. If we introduce the \( 3 \times 3 \) matrices

\[
A' = [a'_1, a'_2, a'_3, a'_4, a'_5, a'_6] \quad B' = [b'_1, b'_2, b'_3, b'_4, b'_5, b'_6]
\]  

(49)

we have

\[
U' = \begin{bmatrix} A' & \bar{A}' \\ B' & \bar{B}' \end{bmatrix} \quad V' = \begin{bmatrix} A'Y & \bar{A}'Y \\ B'Y & \bar{B}'Y \end{bmatrix}
\]  

(50)

where use has been made of eqns (37) and

\[
Y = \begin{bmatrix} 0 & 1 & 0 \\ 1 & \delta & 0 \\ 0 & 0 & 1 \end{bmatrix} = Y'.
\]  

(51)

It is useful to know that

\[
Y^{-1} = \begin{bmatrix} -\delta & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (Y')^T
\]  

(52)

and hence \( Y^{-1} = Y \) when \( \delta = 0 \).

As in the last section the product of \( U \) and \( V \) in eqn (47) can be interchanged. That is

\[
U V^{T} = I
\]  

(53)

or, carrying out the matrix multiplications

\[
A'Y A'^T + \bar{A}'Y \bar{A}'^T = 0 = B'Y B'^T + \bar{B}'Y \bar{B}'^T
\]  

\[
A Y B'^T + \bar{A} Y \bar{B}'^T = I = B Y A'^T + \bar{B} Y \bar{A}'^T
\]  

(54)

This is the modified closure relations for eqns (25). Using the arguments following eqns (25) one is tempted to write
When \( N \) is non-semisimple, i.e. when \( \delta = 0 \), the validity of eqns (55) can be established easily by using the relation [8]

\[
\langle N \rangle \xi_a = \pm \xi_a
\]  

(56)

in which the "\(+\)" sign is for \( a = 1, 2, 3 \), the "\(-\)" sign is for \( a = 4, 5, 6 \), and

\[
\langle N \rangle = \begin{bmatrix} S & H \\ -L & S' \end{bmatrix}
\]  

(57)

Equation (56) certainly applies to \( \xi_1, \xi_2, \) and \( \xi_3, \xi_4, \xi_5, \xi_6 \). Lothe and Barnett [5] and Chadwick and Smith [8] show that it applies to \( \xi_3 \) and \( \xi_4 \) also. Therefore, we have

\[
\langle N \rangle \Psi' = \begin{bmatrix} A' & -A' \\ B' & -B' \end{bmatrix}
\]  

(58)

If we post-multiply both sides by \( V^T \) and use eqn (53), we obtain

\[
H = \frac{i}{\hbar}(A'Y'A'^T - A'Y)^T, \\
L = \frac{i}{\hbar}(B'Y'B'^T - B'Y)^T, \\
S = \frac{i}{\hbar}(A'Y'B'^T - A'YB)^T.
\]  

(59)

Equations (54) and (59) lead to eqns (55).

We will show in the next section that eqns (55) hold also for \( \delta \neq 0 \). In closing this section we point out that to convert eqns (26) to eqns (55) one cannot simply replace \( A, B \), by \( A', B' \). The matrix \( Y \) has to be introduced as shown in eqns (55).

4. CONVERSION FROM THE STROH FORMALISM TO THE MODIFIED FORMALISM

If eqns (55) hold for any \( \delta \), comparison with eqn (26) suggests that the following conversion relations hold:

\[
AA'^T = A'Y'A'^T, \\
BB'^T = B'Y'B'^T, \\
AB'^T = A'YB'^T.
\]  

(60)

We have proved that eqns (25) and hence eqns (60) hold for \( \delta = 0 \). It remains to prove that eqns (60) hold for \( \delta \neq 0 \).

To this end, we will derive the relations between \( \xi'_1, \xi'_2, \) and \( \xi_1, \xi_2 \). Since \( \xi_1, \xi_2, s = 1, 2, \) are scalar multiples of \( \xi_1, \xi_2, \) we let

\[
\xi_1 = \gamma \xi'_1, \quad \xi_2 = \gamma \xi'_2, \\
\eta_1 = \gamma \eta'_1, \quad \eta_2 = \gamma \eta'_2
\]  

(61)

in which eqn (17) has been used and \( \gamma, \delta \) are constants to be determined. From eqns (13) and (36), we have
\[ \xi' = (\phi_2 - \phi_1) \delta \]
\[ \eta' = (\phi_2 - \phi_1) \delta \]
\[ (62) \]

Substituting eqns (61) and (62) into eqns (40), (40)_1 and making use of eqn (19), we obtain
\[ y^2 = -\theta, \quad z^2 = \delta. \]
\[ (63) \]

Recognizing the double solutions for \( y \) and \( z \) in terms of \( \delta \), we let
\[ \alpha = \pm \theta, \quad y^2 = -\delta \]
\[ (64) \]

without identifying which one of the two solutions is for \( y \). Therefore
\[ \xi' = \gamma \xi, \]
\[ \xi'' = \gamma^{-1} (\xi_1 \mp i \xi_2), \]
\[ (65) \]

and \( A' \) from eqn (49), has the expression
\[ A' = [a_1, \gamma^{-1} (a_1 \mp i a_2), a_1]. \]
\[ (66) \]

A similar expression can be written for \( B' \). Let
\[ E = \begin{bmatrix} y & \gamma^{-1} & 0 \\ 0 & \mp i \theta & 0 \\ 0 & \theta & 1 \end{bmatrix}. \]
\[ (67) \]

We then have
\[ A' = AE, \quad B' = BE. \]
\[ (68) \]

By a direct calculation it can be shown that
\[ EYE' = I. \]
\[ (69) \]

Equations (68) and (69) lead to the identities in eqns (60). This completes the proof that eqns (60) and hence eqns (55) hold for any \( \delta \).

With eqns (60) one can convert relations which are valid for simple or semisimple \( N \) to relations for non-semistable or almost non-semi-simple \( N \). For instance, the impedance matrix \( M \) is defined as (1)
\[ M = BA^{-1}. \]
\[ (70) \]

Since
\[ BA^{-1} = (B' B' \bar{A} \bar{B}) (A \bar{B})^{-1}, \]
\[ (71) \]

using eqn (60) we obtain
\[ M = B'A^{-1}. \]
\[ (72) \]

This is one of the few relations for which the conversion is achieved by a simple replacement of \( A, B \) by \( A', B' \).
5. SUM RULES

Several sum rules involving the eigenvalues $\lambda_i$ and eigenvectors $\zeta_i$ have been reported in the literature[3, 14, 15]. The sum rules involve the summations of $\lambda_i$ and $\zeta_i$, which can be written in matrix notation as one of the following[16]

$$\begin{align*}
A^p \lambda^A & , \quad A^p \lambda^B , \quad B^p \lambda^A , \quad B^p \lambda^B \\
A^{p-1} & , \quad A^{p-1} , \quad B^{p-1} , \quad B^{p-1} 
\end{align*}$$

(73)

in which $p$ is an integer, positive or negative, and $P$ is the diagonal matrix

$$P = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}.$$  

(74)

Each of the products in (73) can be expressed in terms of the real matrices $H_i$, $L_i$, $S_i$ and $N_i$, $i = 1, 2, \ldots$. By a direct calculation, it can be shown that

$$E^p E^{-1} = P,$$  

(75)

in which $E$ is given in eqn (67) and, by eqn (69)

$$E^{-1} = Y E^T = \begin{bmatrix} \gamma^{-1} & \mp \eta^{-1} & 0 \\ 0 & \pm \eta & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$  

(76)

$$P = \begin{bmatrix} p_1 & 1 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}.$$  

(77)

We see that $P$ is the Jordan canonical matrix when $p_1 = p_i$. From eqns (68), (69) and (75) we have the following conversion relations:

$$\begin{align*}
A^p \lambda^A & = A^p \gamma \lambda^A Y^{-1} \\
A^p \lambda^B & = A^p \gamma \lambda^B Y^{-1} \\
B^p \lambda^A & = B^p \gamma \lambda^A Y^{-1} \\
B^p \lambda^B & = B^p \gamma \lambda^B Y^{-1} 
\end{align*}$$

(78)

and

$$\begin{align*}
A^p \lambda^{-A} & = A^p \gamma \lambda^{-A} Y^{-1} \\
A^p \lambda^{-B} & = A^p \gamma \lambda^{-B} Y^{-1} \\
B^p \lambda^{-A} & = B^p \gamma \lambda^{-A} Y^{-1} \\
B^p \lambda^{-B} & = B^p \gamma \lambda^{-B} Y^{-1} 
\end{align*}$$

(79)

Equations (78) suggest that $P^T Y$ is symmetric. Indeed, it is readily shown that
\[ P^* = \begin{bmatrix} p_1 & y_1 & 0 \\ 0 & p_2 & 0 \\ 1 & 0 & p_3 \end{bmatrix} \]

where

\[ y_1 = \sum_{k=1}^3 p_1^{k-1} \cdot p_1^{k-1} \]

\[ = \begin{cases} (p_2 - p_1) / (p_2 - p_1), & \text{if } p_2 \neq p_1, \\ \eta p_1^{-1}, & \text{if } p_2 = p_1. \end{cases} \]

Hence

\[ P^* Y = \begin{bmatrix} p_1 & y_1 & 0 \\ p_2 & 0 & 0 \\ 0 & 0 & p_3 \end{bmatrix} = (P^*Y)^T. \]

6. SEPARATION OF \( \xi \) INTO \( a_i \) AND \( b_i \)

The Stohr eigen-relation was in fact based on the earlier version, eqns (5) and (8) proposed by Eshelby et al. Thus instead of finding the 6-vector \( \xi \) from eqn (11) one could find the 3-vectors \( a \) and \( b \) from eqns (5) and (8). This may have some advantages in a numerical calculation because not only the matrix \( B \) is smaller than \( N \), one does not have to find the inverse \( T^{-1} \) as shown in eqns (13). When \( N \) is non-semisimple, so is \( D(p) \) of eqn (5). This means that when \( p_1 = p_2, a_i = a_i \). To modify eqns (5) and (8) for the cases when \( D(p) \) is non-semisimple or almost non-semisimple, we follow the derivation of eqns (32). We obtain

\[ D(p_1) a_i = \theta \]

\[ D(p_2) a_i + [(R + R') + (p_2 - p_2)T] a_i = 0 \]

(83)

As in the modification of eqn (8), we have

\[ b_i = (R' + p_2 T) a_i = \begin{bmatrix} \frac{1}{p_2} Q + R \end{bmatrix} a_i \]

\[ b_i = (R' + p_2 T) a_i + Ta_i = \begin{bmatrix} \frac{1}{p_2} Q + R \end{bmatrix} a_i + \frac{1}{p_2} Q a_i \]

(84)

Equations (83) and (84) provide \( a_i, b_i, b_i, b_i \) and \( b_i \) which form the components of \( \xi \) and \( \xi \). One then finds \( q_i', q_i' \) from eqns (37) and orthonormalize the eigenvectors as outlined in Section 3. To compose the system, one finds \( a_i, b_i, b_i, b_i \) from eqn (17) and normalize \( \xi \) using eqn (19).

For isotropic materials we have \( p_1 = p_2 = \eta, a_i = a_i \), and the outlined procedure leads to

\[ A' = \begin{bmatrix} k_1 & -\eta k_1 & 0 \\ \eta k_1 & -k_1 & 0 \\ 0 & 0 & k_2 \end{bmatrix} \]

\[ B' = \begin{bmatrix} 2k_1 & k_1 & 0 \\ -2k_1 & -k_1 & 0 \\ 0 & 0 & ik_2 \end{bmatrix} \]

(85)

(86)
\[ k_1 = \frac{1}{3(1-\nu)}, \quad k_2 = -\frac{1}{2\mu} + \frac{(1-\nu)^2}{4\mu}, \quad \kappa = \frac{3-4\nu}{2}. \]  

(87)

where \( \mu \) and \( \nu \) are, respectively, the shear modulus and Poisson's ratio. With \( A', B' \) given by eqns (85), (86) and \( V \) by eqn (51) with \( \delta = 0 \), eqns (55) provide \( H, L, S \) for isotropic materials. The non-zero elements of \( H, L, \) and \( S \) are

\[
H_{11} = H_{12} = \frac{3-4\nu}{4\mu(1-\nu)}; \quad H_{13} = \frac{1}{\mu} \\
L_{11} = L_{12} = \frac{\mu}{1-\nu}; \quad L_{13} = \mu \\
S_{11} = -S_{12} = \frac{1-2\nu}{2(1-\nu)}
\]

(88)

This agrees with the results obtained by using the integral formalism in Ref. [8].

7. CONCLUDING REMARKS

The modified sextic formalism presented here applies to any matrix \( N \) which is simple, almost non-semisimple or non-semisimple. The formalism is particularly useful when \( N \) is non-semisimple or almost non-semisimple. Thus instead of the integral formalism[8], eqns (55) offer an alternate way of obtaining the three real matrices \( H, L, S \) when \( N \) is non-semisimple or almost non-semisimple.

We did not consider the possibility of a non-semisimple \( N \) in which \( p_1 = p_2 = p_3 \), and \( \xi = \xi = \xi = \xi = \xi = \xi \). We have not seen such an example and it appears unlikely that there exists a real material which leads to \( p_1 = p_2 = p_3 \), and \( \xi = \xi = \xi = \xi = \xi = \xi = \xi \). For isotropic materials \( p_1 = p_2 = p_1 = p_3 \), but \( \xi = \xi = \xi = \xi = \xi = \xi = \xi \), and hence the modified formalism applies to isotropic materials as shown in the last section.

REFERENCES