UNIFORM HEAT FLOW DISTURBED BY AN ELLIPTICAL RIGID INCLUSION EMBEDDED IN AN ANISOTROPIC ELASTIC MATRIX

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INTRODUCTION

For a homogeneous body that deforms freely and is governed by the assumptions of steady-state linear anisotropic elasticity, the linear distribution of temperature will not induce stresses inside the body. However, if the body contains cracks, holes, or inclusions, the temperature field will be disturbed and thermal stresses will be induced. Severe thermal stresses even arise near crack tips. The thermal stress concentration induced by holes or cracks has been studied by several investigators. Detailed literature survey may be found in a recent paper by Hwu [1]. Nevertheless, very few studies have been done for inclusion problems, which are also important for many practical applications such as bolted joints and fiber-reinforced composites. By using the complex variable formulation, Chen [2] studied the problem of an elliptical elastic inclusion under uniform heat flow. However, due to the mathematical complexity, a set of simultaneous algebraic equations need to determine the unknown coefficients is left to the readers. If the inclusions have sharp edges, large stresses arise in the vicinity of these edges. Related problems such as insulated or conductive ribbon-like rigid inclusions in an infinite isotropic medium have been studied by Sekine [3] and Sekine and Mura [4].

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Laser, Sumi [5, 6] combined the complex variable method with the mapping-collocation technique to solve the same problems but with finite isotropic and anisotropic plates.

In this paper, an explicit closed form solution for the thermal stress induced by an insulated elliptical rigid inclusion embedded in an anisotropic elastic matrix has been obtained by applying the extended version of Stroh’s formalism for anisotropic thermoelasticity [1, 8]. During the derivation, several identities relating the material properties to the complex eigenvectors are used to simplify the formulas. By letting the minor axis of the ellipse approach to zero, the solution for a rigid line inclusion can be obtained. The strength of thermal stress singularity is defined in a similar way to the stress intensity factor of crack problems and is found to be dependent on the material properties. Since the present formulation is based upon the assumption that the material eigenvalues are distinct, in order to be applicable to cases where the eigenvalues are repeated, such as for isotropic materials, a small perturbation of the material constants is introduced; the results are in good agreement with those given by Sjadi [5].

PLANE ANISOTROPIC THERMOELASTICITY

Based upon the Stroh’s formalism in anisotropic elasticity, a simple and compact version of general solution for the uncoupled steady-state anisotropic thermoelasticity can be expressed as [1]

\[
\begin{align*}
T &= 2 \Re \left( g(x, z) \right) \\
u &= 2 \Re \left( \frac{k_1 + ik_2}{q + ik} \right) \\
\phi &= 2 \Re \left( \frac{B(\zeta, z)}{q + ik} \right)
\end{align*}
\]

(16)

where

\[
A = [a_1, a_2, a_3], \quad B = [b_1, b_2, b_3]
\]

\[
x_1 = x_1 + r \xi_2, \quad x_2 = x_2 + r \xi_3
\]

(17)

The superscript $T$ denotes the transpose. $C_{ijkl}$, $\xi_1$, and $\xi_2$ are the elastic constants which are assumed to be fully symmetric and positive definite so that the strain energy is positive. $r$ and $q(x, z)$ are the heat eigenvalues with positive imaginary part and the associated generalized eigenvectors of

\[
\begin{align*}
\frac{d^2}{dr^2} + 2k_2 \frac{d}{dr} + k_1 &= 0 \\
Nq &= r^2q + \gamma
\end{align*}
\]

where

\[
\gamma = \left[ \begin{array}{c} \gamma \end{array} \right], \quad N = \left[ \begin{array}{c} N_1 \\ N_2 \end{array} \right], \quad \beta = \left[ \begin{array}{c} \beta_1 \\ \beta_2 \\ \beta_3 \end{array} \right]
\]

(20)

$\beta_1$, $\beta_2$, and $\beta_3$ are the thermal moduli which are assumed to be symmetric. In Eq. (20) the symmetry assumption of $k_1$ has been employed. Note that the general solution given in Eq. (1) is obtained under the assumption that the heat eigenvalue and the elasticity eigenvalues are distinct. For cases where they are repeated, a small perturbation of the material constants can be employed to avoid the degenerate problem; otherwise, a modified solution should be applied [6]. However, if the final solutions do not contain the eigenvectors $a_1$, $b_1$, and $c$, the problems of repeated eigenvalues can then be avoided, which can usually be done through the use of identities given in the following.

Due to the orthogonality relation among the eigenvectors $\xi_j$ derived by Stroh [9], three real matrices have been introduced as

\[
S = (2AB^T - I), \quad H = 2AAB^T, \quad L = -2\Omega B^T
\]

(22)
in which $I$ is the unit matrix, $H$ and $L$ are symmetric and positive definite, and $S$, $E$, $H^{-1}$, $H^{-1}$ are anti-symmetric. An identity related to the thermostatic properties and used in this paper for the simplification of formula is [1]:

$$\text{Sc} + \text{Hd} = \text{ie} + \sum\tilde{\gamma}$$

$$-\text{Le} + \sum\text{S'd} = \text{id} + \tilde{\gamma}$$

(5a)

where

$$\tilde{\gamma} = -\int_{0}^{1} \cos \theta \, \gamma_{i}(\theta) \, d\theta$$

$$\gamma_{i}(\theta) = -N_{i}(\theta) \beta \text{m}(\theta)$$

(5b)

and

$$\text{m}(\theta) = (\cos \theta \, \sin \theta \, \cos \theta \, \sin \theta)^{T}$$

$$\beta = [\beta_{1}, \beta_{2}, \beta_{3}]$$

$N_{i}(\theta), i = 1, 2, 3$, are the generalized form of $N_{i}$ [1].

FIELD SOLUTIONS

Consider a rigid inclusion embedded in an infinite isotropic plate under a uniform heat flux $h_{0}$ directed at an angle $\psi$ with the positive $x_{1}$-axis. The contour of the inclusion is considered to be an ellipse, of which the boundary is represented by

$$x_{1} = a \cos \psi \quad x_{2} = b \sin \psi$$

(6)

where $a, b$ are the half lengths of the major and minor axes of the ellipse and $\psi$ is a real parameter. If the inclusion is considered to be perfectly bonded with the matrix and is insulated, the boundary conditions for this problem can be written as

$$h_{0} = 0 \quad \text{u} = u_{m} \quad \text{along the inclusion boundary}$$

(7a)

where $h_{0}$ is the heat flux in the direction of $m$ which is normal to the surface of the inclusion and $u_{m}$ is the rigid body displacement caused by the relative rotation $\omega$ between the matrix and inclusion, which can be written as

$$u_{m} = x_{1} \omega_{1} + x_{2} \omega_{2}$$

$$\omega_{1} = \omega_{2} = \omega(0)$$

(7b)

The expressions for $m$ and $u_{m}$ are shown in Eq. (5c) where the angle $\theta$ is directed counter-clockwise from the positive $x_{1}$-axis to the direction of $m$. The relation between $\theta$ and $\psi$ is

$$\phi = \cos \theta = \sin \psi$$

$$\psi = \sqrt{a^{2} \sin^{2} \phi + b^{2} \cos^{2} \phi}$$

(7c)

At infinity the stresses $\sigma_{0}$ approach zero and the heat flux $h_{1}$ tends to be a constant value, which can be expressed as

$$\sigma_{0} = 0 \quad h_{1} = h_{0} \sin \psi$$

(8)

If the rigid inclusion is considered to have nonzero thermal conductivity, the temperature and heat flux across the boundary should be continuous. The boundary conditions along the interface between the inclusion and matrix may then be replaced by $T = T_{0}$, $h_{0} = h_{0}^{*}$, $u = u_{m}$, where the superscript * refers to the rigid inclusion. In the following, only an insulated rigid inclusion will be studied.

To find a solution satisfying the conditions given in Eqs. (7) and (8), the choice of arbitrary functions $g(x)$ and $f(x)$ becomes critical in the solution procedures.

Since a certain analogy exists between the hole and rigid inclusion problems, one may use the result for the hole problem [1] as a reference. After referring to the solution for hole problems, the general solution for rigid inclusion problems can be written as

$$u = \sum_{k_{1}} \text{Re}(A_{k_{1}}) \text{e}^{k_{1}x_{1}} + \sum_{k_{2}} \text{Re}(A_{k_{2}}) \text{e}^{k_{2}x_{1}} + 2 \sum_{k_{1}} \text{Re}(A_{k_{1}}) \text{B}_{k_{1}} \text{e}^{k_{1}x_{1}} + 2 \text{Re}(g(x)))$$

(9a)

$$f_{x}(z) = \sum_{k_{2}} \text{Re}(B_{k_{2}}) \text{e}^{k_{2}x_{1}} + 2 \sum_{k_{1}} \text{Re}(B_{k_{1}}) \text{B}_{k_{1}} \text{e}^{k_{1}x_{1}} + 2 \text{Re}(g(x)))$$

(9b)

where $g(z)$ and $f_{x}(z)$ are chosen to be

$$g(x) = g_{0}^{*} + \frac{a^{2}}{a + b} \left( \frac{a^{2}}{a + b} - \frac{1}{a^{2}} \right) \left( \frac{\sqrt{a^{2} + b^{2}}}{a} \right) + \frac{a^{2}}{a + b} \left( \log \left( \frac{a}{\sqrt{a^{2} + b^{2}}} \right) - \frac{a^{2}}{a + b} \right)$$

$$f_{x}(z) = \frac{1}{a + b} \left( \frac{a^{2}}{a + b} - \frac{1}{a^{2}} \right) \left( \frac{\sqrt{a^{2} + b^{2}}}{a} \right) + \frac{a^{2}}{a + b} \left( \log \left( \frac{a}{\sqrt{a^{2} + b^{2}}} \right) - \frac{a^{2}}{a + b} \right)$$

(9c)

$$f_{x}(z) = \frac{a^{2}}{a + b} \left( \log \left( \frac{a}{\sqrt{a^{2} + b^{2}}} \right) - \frac{a^{2}}{a + b} \right)$$

(9d)
and \( q_{1a}, q_{1b}, k = 1, 2, 3 \), are \( 3 \times 1 \) real column matrices which are related to the complex column matrix \( q_i \) by:

\[
q_i = A^*q_{1a} + B^*q_{1b}
\]

Note that the choice of \( g(z) \) and \( f(z) \) given in Eq. (96) is almost the same as that of the hole problem shown in [11] except that the terms associated with \( z^2 \) and \( z^3 \) are now included in order to account for the uniform heat flux at infinity.

In hole problems, due to the application of superposition principle, a negative uniform heat flux is considered to be applied on the hole boundary. However, it is not appropriate in rigid inclusion problems since the rigid body motion conditions of inclusions cannot be satisfied.

The problem now reduces to the determination of the unknown complex constants \( g_i, f_i, k \) and real constants \( q_{1a}, q_{1b} \), which should satisfy the boundary conditions shown in Eqs. (7) and (8). Substituting Eq. (96), into the general solution of heat flux \( A_i \) shown in Eq. (102), and using the boundary condition (80) of heat flux at infinity, we obtain:

\[
\varepsilon_i = -\frac{h_b}{4k\eta}(\cos \phi \pm \sin \phi)
\]

(10a)

where \( k \) is a real constant defined as:

\[
k = k_2(\tau - \gamma)/2i
\]

(10b)

To apply the infinity conditions of stresses, Eq. (80), we evaluate the stresses by \( \sigma_{11} = -\sigma_{01}, \sigma_{02} = \sigma_{22} \), and Eq. (99a). Since \( g(z) = 2g_0z, f(z) = z^2z \), \( f(z) = 0 \) at \( z = z_0 \) or \( z = \infty \), we have, with the aid of the following identities [10]:

\[
\begin{align*}
2B^* & = iL, \quad 2B^* = iL, \\
2B^*(\rho_2) & = N_{11} - i(N_{11}^*S + N_{11}^*H), \\
2B^*(\rho_2^2) & = N_{11}^* - i(N_{11}^*S + N_{11}^*L), \\
2B^*(\rho_2^3) & = -i(N_{11}^*N_{11}N^*_{11} - N_{11}^*N^*_{11}S - i(N_{11}^*N_{11}^*S + N_{11}^*N^*_{11}L)) \\
2B^*(\rho_2^4) & = (N_{11}^*N_{11}^*S - N_{11}^*N^*_{11})L
\end{align*}
\]

(11)

Using the infinity conditions, the constants \( \varepsilon_i, q_{1a}, q_{1b} \), and \( q_i \), have been determined in Eqs. (10) and (11). The remaining unknown constants \( q_{1a} \) and \( q_{1b} \) can be found by the use of the inclusion boundary conditions shown in Eq. (7a). By using the coordinate transformation and applying Eq. (10a) and (3a), we have:

\[
h_b = 2k \text{ Im}(\tau + \sin \phi \pm \cos \phi g(z))
\]

(14)

Where \( \text{Im} \) stands for the imaginary part of a complex number. Substituting Eq. (96b), into Eq. (14) and carefully differentiating \( g(z) \) with respect to its argument
twice \([1]\), the condition for the inserted inclusion (3a), provides that
\[
\xi_2 = \frac{-b}{2H} \left( d \cos \phi + e \sin \phi \right) \tag{15}
\]

We now consider the displacement continuity conditions which lead to \(\eta_a = -\eta_b\). By a careful differentiation with respect to \(\eta\) \([1]\) and the use of identities which are similar to those shown in Eqs. (1) for \(A'\), \(A\left(p_1\right)A'\), \(A\left(p_2\right)B\), \(A\left(p_2\right)A'\), \(A'\left(p_1^2\right)B\) \([10]\), the continuity conditions give us
\[
b^2 \eta_b \left[ N_c N_2 + N_c N_3 \right] \eta_{a2} - b^2 \left[ N_c N_2 + N_c N_3 \right] \eta_{a3} - \frac{b^2}{2} \left[ \eta_b \left( e \cos \phi - d \sin \phi \right) \right] = 0
\]
\[
\frac{b}{2} \left( N_c N_2 - N_c N_3 - N_c N_1 \right) \eta_{a2} - \frac{b}{2} \left( N_c N_2 - N_c N_3 + N_c N_1 \right) \eta_{a3} - \frac{b}{2} \left( N_c N_2 + N_c N_3 \right) \eta_{a3} = 0
\]
\[
2 a b \left( N_c N_2 \eta_{a2} + N_c N_3 \eta_{a3} - \frac{a}{2} H - b N_c \right) \eta_{a2} = 2 a b \left( N_c N_2 \eta_{a2} + N_c N_3 \eta_{a3} + \frac{a}{2} H - b N_c \right) \eta_{a2} = 0
\]
\[
\frac{a}{2} H - b N_c \eta_{a2} + \frac{a}{2} H - b N_c \eta_{a2} + 2 b H \left( e \cos \phi - d \sin \phi \right) = 0
\]

Due to the multivalued characteristics of logarithmic functions contained in \(g(x)\) and \(f(x)\), the requirement of singularity of stress function provides
\[
\left( a S^2 + b N_c^2 \right) \eta_{a2} + (-a L + b N_c) \eta_{a3} + 2 b H \left( e \cos \phi - d \sin \phi \right) = 0
\]
\[
\phi_{x2} = \frac{-b}{2H} \left( d \cos \phi + e \sin \phi \right) \tag{17}
\]

We now have four equations to solve for four unknown real vectors \(\eta_{a2}, \eta_{a3}\) and \(\eta_{a2}, \eta_{a3}\). By a similar mathematical manipulation used in \([1]\), the solutions are obtained as
\[
\begin{align*}
\eta_{a2} &= (a N - b N_c)^{-1} \frac{2 d}{2 H} \left( -e \cos \phi + d \sin \phi \right) \tag{18a} \\
\eta_{a3} &= \frac{2 d}{2 H} \left( e \cos \phi - d \sin \phi \right) \tag{18b}
\end{align*}
\]

where
\[
\begin{align*}
\xi &= -2 \operatorname{Re} \left( g(x) e^{-i \phi} \right) \frac{1}{2} \left( a \cos \phi + b \sin \phi \right) \tag{18c} \\
\phi &= \left( -a \left( N_2 + N_3 \right) \eta_{a2} - \left( a^2 + b^2 \left( N_2 + N_3 \right) \right) \eta_{a3} \right) \tag{18d}
\end{align*}
\]
and \(N\) is defined as
\[
N = \frac{1}{\sqrt{g}} \eta_a \quad \eta_a = \left( N - H \right) \tag{18e}
\]

The existence of the inverse of \(N - H\) is discussed in \([1]\).

If one is interested in the interfacial stresses along the inclusion boundary, calculation may be performed by using the field solution of the matrix. The stress components based upon the coordinate system \((\xi, \eta)\) are obtained as
\[
\begin{align*}
\sigma_{\eta\eta} &= -a \left( \xi \right) \phi_{x2} \\
\sigma_{\xi\eta} &= -a \left( \xi \right) \phi_{x3} \\
\sigma_{\xi\xi} &= -a \left( \xi \right) \phi_{x3} \\
\sigma_{\eta\eta} &= -a \left( \xi \right) \phi_{x3} \\
\phi_{x2} &= \frac{-b}{2H} \left( d \cos \phi + e \sin \phi \right)
\end{align*}
\]

By a careful differentiation with respect to \(\eta\) or \(m(\xi)\) and the use of identities such as Eqs. (4), (5), (11) and the property that the product of \((N)\) and \((\xi)\) commutes \([13]\), i.e.,
\[
\eta_a N(\xi) = \xi N(\eta) \quad \xi N(\eta) \quad \eta_a N(\xi)
\]
once may obtain
\[
\phi_{x2} = \frac{-b}{2H} \left( d \cos \phi + e \sin \phi \right) \frac{1}{2} \left( a \cos \phi + b \sin \phi \right) \tag{18a}
\]
\[
\begin{align*}
\eta_{a2} &= \left( a N - b N_c \right)^{-1} \frac{2 \eta_{a2}}{2 H} \left( -e \cos \phi + d \sin \phi \right) \tag{18b} \\
\eta_{a3} &= \frac{2 \eta_{a2}}{2 H} \left( e \cos \phi - d \sin \phi \right) \tag{18c}
\end{align*}
\]
for crack problems, we may define the strength of thermal stress singularity as

\[
\begin{align*}
\left\{ f_a \right\} & = \lim_{r \to a} \left( \frac{r}{s} \right) \left\{ \sigma_a \right\} \\
\left\{ f_b \right\} & = \lim_{r \to b} \left( \frac{r}{s} \right) \left\{ \sigma_b \right\}
\end{align*}
\]

Substituting Eq. (23) into Eq. (24), using the results obtained in Eqs. (22), and applying the identities (5), we obtain

\[
\begin{align*}
\left\{ f_a \right\} & = \frac{2\pi \nu}{R} \left( \nu \sigma_a - \sigma_b \right) \\
\left\{ f_b \right\} & = \frac{2\pi \nu}{R} \left( \nu \sigma_b - \sigma_a \right)
\end{align*}
\]

\[
\begin{align*}
2 \nu + 4 \pi \frac{\sigma_a}{R} - 2 \nu \frac{\sigma_b}{R} - 2 \nu \frac{\sigma_a}{R} + 4 \pi \frac{\sigma_b}{R} & = 2 \nu + 4 \pi \frac{\sigma_a}{R} - 2 \nu \frac{\sigma_b}{R} - 2 \nu \frac{\sigma_a}{R} + 4 \pi \frac{\sigma_b}{R}
\end{align*}
\]

RESULTS AND DISCUSSION

All the solutions derived in this paper are obtained under the assumption that the heat eigenvectors \( \alpha \) and the elasticity eigenvectors \( \beta_a, a = 1, 2, 3, \) are distinct. If the final solutions do not contain the eigenvectors \( \alpha, \beta_a, \beta_b, \) such as the hoop stress and the stress intensity factor shown in How's paper [1], the solutions of repeated eigenvectors may be avoided. However, the general solutions shown in Eqs. (13), (18), (20), (22), (23), and (25) do contain the eigenvectors. In order to apply the present results to degenerate materials like isotropic materials, a small perturbation of the material constants is employed to obtain distinct eigenvectors. An example of isotropic materials with rigid line inclusions has been done by this perturbation technique, in which the material constants are chosen as

\[
E_{11} = E_{22} = E_{33} = 200 \text{ GPa}
\]

\[
\nu_{12} = \nu_{23} = \nu_{31} = 0.28 + \delta
\]

\[
G_{12} = G_{23} = 78.135 \text{ GPa}
\]

\[
\sigma_1 = \sigma_2 = \sigma_3 = 28.1 \times 10^4 \text{ m/K}
\]

\[
\kappa_1 = 4.62 \text{ Nt/m/K} \quad \kappa_2 = 4.62 + \delta \text{ Nt/m/K}
\]

where \( \delta \) is the small perturbed number and \( \alpha_1 \) are the coefficients of thermal expression which are related to the thermal mobilities \( \beta_1 \) by \( \beta_1 = C_{11} / \alpha_1 \). The numerical calculation for the strength of thermal stress singularity given in Eq. (25) shows that when the heat flow is in the directions of \( \varphi = 0^\circ, 45^\circ, 90^\circ \), respectively, \( F(\varphi) = 0.010(0.005040.01130.10.157) / s - e - \alpha_1 / \kappa_2 / \delta \), which are almost identical to those shown by Sekine [3]. Note that \( \delta \) is chosen in the range of
$10^{-13} - 10^{-15}$, and the error is in the range of $10^{-13} - 10^{-15}$ depending on the small pictured number $\theta$ chosen in calculation.

As for general anisotropic materials, we consider Graphite/Epoxy AS/3501 of which the material constants are

\begin{align*}
E_{||} &= 144.8 \text{ GPa} & E_{\perp} &= E_{\perp} = 9.7 \text{ GPa} \\
\nu_{||} &= \nu_{\perp} = \nu_{\perp} = 0.3 \\
G_{||} &= G_{\perp} = G_{\perp} = 4.1 \text{ GPa} \\
\alpha_{||} &= -0.3 \times 10^{-6} \text{ m/mK} & \alpha_{\perp} &= \alpha_{\perp} = 28.1 \times 10^{-6} \text{ m/mK} \\
k_{||} &= 4.63 \text{ Ns/mK} & k_{\perp} &= k_{\perp} = 0.72 \text{ Ns/mK}
\end{align*}

Figs. 1, 2, and 3 show the contour diagrams for the temperature distribution near the rigid inclusion. The temperature is nondimensionalized by $T/T_{0}$, $\theta$ is the angle between the fiber and $x_3$-axis. The uniform heat flow applied at infinity can be seen by the equidistant and parallel lines shown in these figures. However, the normal of these parallel lines is not necessarily the flow direction such as in Figs. 2 and 3, which can be understood by the relation $k_{||} = -k_{||}T_{0}$. For the case that $\theta = 0^\circ$, $\phi = 0^\circ$ (Fig. 1), we have $k_{||} = k_{\perp} = k$, $k_{\perp} = 0$. Hence, $k_{||} = -k_{||}T_{0}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Temperature distribution around the rigid inclusion embedded in the orthotropic matrix ($\theta = 0^\circ$, $\phi = 0^\circ$, $\lambda/\mu = 0.5$).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Temperature distribution around the rigid inclusion embedded in the orthotropic matrix ($\theta = 0^\circ$, $\phi = 45^\circ$, $\lambda/\mu = 0.5$).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{Temperature distribution around the rigid inclusion embedded in the orthotropic matrix ($\theta = 0^\circ$, $\phi = 45^\circ$, $\lambda/\mu = 0.5$).}
\end{figure}
where the minor axis of the ellipse is made to be zero, and anisotropic media in which the material eigenvalues are repeated and a small perturbation of the material constants is introduced.

REFERENCES


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