THERMOELASTIC INTERFACE CRACK PROBLEMS
IN DISSIMILAR ANISOTROPIC MEDIA

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Abstract—By applying the extended version of Stroh’s formulation and a special technique of
analytical continuation, a general solution for the thermoelastic collinear interface cracks between
dissimilar anisotropic media has been obtained in this paper. Spatial localization and examples are
given for the application of the final results to the whole-field solutions of the temperature, heat
flux, displacements and strains. The explicit expressions for the interface stresses and crack opening
displacements are also provided. Based upon this general solution, two special examples of single
crack problems are solved explicitly. One is homogeneous media, the other is in anisotropic layers.
The former is studied for the purpose of verifying the detailed calculation, verifying the general
solution and discussing the validity of the solution. A closed form solution is obtained for the latter
case subjected to a uniform heat flux and bending.

1. INTRODUCTION

One of the most frequently encountered problems in composite laminates is interface
cracking, sometimes also known as delamination. Delaminations in layered composite
materials may occur due to a variety of reasons, such as low energy impact, manufacturing
defects or high stress concentrations at geometry or material discontinuities (Kardomateas
and Schmaeuer, 1988). Due to recent aerospace and commercial applications, laminated
composites are experiencing an increased utilization in high temperature environments.

A quantitative assessment of the effect of realistic delaminations on the strength and
life-time of a laminate is difficult. Consequently, analytical efforts to date have only attempted
to quantify the effect of idealized delaminations. Williams (1959) discovered the so-called
oscillatory near tip behavior for an interface crack between two isotropic materials. Since
then, many authors have discussed the interface crack problems such as England (1965),
 Erdogan (1965), Rice (1988) and Sau (1989) for isotropic media; and Getoh (1967),
Clemens (1971), Willis (1971), Ting (1986), Bassani and Qu (1989), Qu and Bassani (1989),
Sau (1990a), Wu (1990), Gao et al. (1992) and Hwu (1991) for anisotropic media. By a
skill of orthotropy excising (Sau, 1990b) which reduces plane elasticity problems for
orthotropic materials to equivalent problems for materials with cubic symmetry, certain
interface crack solutions for orthotropic materials may also be obtained directly from
isotropic solutions. In spite of the vast resources for the interface crack problems, very few
published analytical studies are available for the thermoelastic interface crack problems.

The steady state thermoelastic problems of interface cracks between dissimilar isotropic
media have been studied by Erdogan (1965), Barber and Comninou (1982, 1983), Martin-
Moran et al. (1983) and Suma and Ueda (1990). As for the cracks between dissimilar
anisotropic media, solutions were presented by Clemens (1983) through the use of the
complex variable method. Due to the lack of identities among the thermoelastic constants
developed in later years (Ting, 1988; Hwu, 1990), the solutions provided by Clemens
(1983) are complicated. Moreover, without further notification these solutions can only be
applied to interface, not the full field domain.

In this paper, we consider an arbitrary number of collinear cracks lying along the
interface subjected to an arbitrary and self-equilibrated loading and heat flux on the upper
and lower surfaces of the cracks. The materials are assumed to be perfectly bonded at all
points except those lying in the region of cracks. To solve this problem, an extended version
(Hwu, 1990) of Stroh’s formalism (Stroh, 1958) for plane anisotropic thermoelasticity is
applied. A special technique of analytical continuation (Muskhelishvili, 1954) which is
similar to the one proposed by Suo (1990a), is also developed to consider the misfit of material constants along the interface. In Hwu's (1990) paper, the cracks are embedded in the homogeneous anisotropic media. While in Suo's (1990a) paper, no thermal effect is considered for the cracks hing along the interface of two dissimilar anisotropic media. The combination of the extended version of Strehl’s formalism with the method of analytical continuation leads the problems of thermoelastic interface cracks to a Hilbert problem of vector form. A derivation of the general solution to the vector form Hilbert problem is provided in this paper. An explicit closed form solution for the thermoelastic interface crack problems is therefore obtained, which is expressed in compact matrix notation.

Special cases and examples are given for the application of the final results to the whole field solutions of temperature, heat flux, displacements and stresses. The explicit expressions for the interface stresses and crack opening displacements are also provided. The solutions are valid only when the assumption of fully open crack is not violated. If this is violated, a partial contact crack may be assumed which is similar to the problems discussed by Marín-Moran et al. (1983) and Barber and Comninou (1983). Ting (1986) showed that if the negative imaginary part W of the bimaterial matrix M (the definition can be found in this paper) is identical to zero, there will be no interpenetration problem for interface cracks. However, it has been argued by Roe (1988) that the solutions can still be used to characterize the interface fracture process since the contact zone size is found to be extremely small for a broad range of bimaterial and loading configurations of practical importance. Hence, in this paper no detailed validation of the validity of fully open crack assumption is provided. The main concern is devoted to the derivation of the general solution to the problems of thermoelastic interface collinear cracks without any restriction to the material properties of anisotropic media.

Two typical examples are solved explicitly. The simplest case when the two media are composed of the same material is studied for the purpose of presenting the detailed calculation and verifying the general solutions. A closed form solution is obtained for the case of bimaterial interface cracks subjected to uniform heat flux and loading.

2. PLANE ANISOTROPIC THERMOELASTICITY

2.1. General solutions

Based upon Strehl’s formalism (Strehl, 1958) in anisotropic elasticity, a simple and compact version of general solutions for the uncoupled steady-state plane anisotropic thermoelasticity has been presented by Hwu (1990) as

\[ T = 2 \text{ Re} \left\{ \sigma(z_1) \right\}, \quad \phi = 2 \text{ Re} \left\{ \sum_{l=1}^{\infty} b_l \phi(z_l) + \text{d}(\hat{z}_l) \right\}, \quad \sigma = 2 \text{ Re} \left\{ \sum_{l=1}^{\infty} a_l \phi(z_l) + \text{e}(z_l) \right\}, \quad \phi_{1} = -\phi_{2}, \quad \sigma_{1} = -\phi_{1}, \quad (1a) \]

where

\[ z_1 = x_1 + p_1 x_2, \quad z_2 = x_1 + \lambda x_2, \quad (1b) \]

\((x_1, x_2)\) is a fixed rectangular coordinate system. Re stands for the real part of a complex number. The prime () denotes differentiation with respect to its argument. A comma stands for differentiation. \(T, a, b, \phi, \sigma_1\) and \(\sigma_2\) represent, respectively, the temperature, heat flux, displacements, stress functions and stresses. \(x_1\) are the heat conduction coefficients. \(f_1(z_2), \quad \alpha = 1, 2, 3\) and \(g_2(x_2)\) are arbitrary functions with complex arguments \(z_1\) and \(z_2\), respectively. \(p\) and \((a_1, a_2)\) are the elasticity eigenvalues with positive imaginary parts and the associated eigenvectors of

\[ \mathbf{N}_2 = \rho \mathbf{c}, \quad (2a) \]

where
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\[ \mathbf{N} = \begin{bmatrix} N_1 & N_2 \\ N_1 & N_2 \end{bmatrix}, \quad \mathbf{\xi} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \quad (2b) \]

\[ N_1 = T^{-1}R^T, \quad N_2 = T^{-1} = N_1, \quad N_3 = RT^{-1}R^T - Q = N_1, \quad (2c) \]

and

\[ Q_0 = C_{ijkl}, \quad R_0 = C_{ijkl}, \quad T_0 = C_{ijkl}. \quad (2d) \]

The superscript T denotes the transpose. \( C_{ijkl} \) are the elastic constants which are assumed to be fully symmetric and positive definite so that the strain energy is positive. \( \tau \) and \( (c, d) \) are the heat eigenvalue with positive imaginary part and the associated generalized eigenvectors of

\[ k_{12} \tau^2 + 2k_{12} \tau + k_{11} = 0, \quad \mathbf{N}_0 = \tau \mathbf{q} + \tau \mathbf{y}, \quad (3a) \]

where

\[ \mathbf{y} = \begin{bmatrix} 0 & N_2 \\ 1 & N_1 \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix}, \quad (3b) \]

and

\[ (\mathbf{f}_{ij}), = \mathbf{f}_{ij}, \quad (\mathbf{g}_{ij}) = \mathbf{g}_{ij}. \quad (3c) \]

\( \beta_1 \) are the thermal moduli which are assumed to be symmetric. In eqn (3a), the symmetry assumption of \( k_{ij} \) has been employed. Note that the general solution given in (1) is obtained under the assumption that the heat eigenvalue and the elasticity eigenvalues are distinct. If they are repeated, a small perturbation of the material constants can be employed to avoid the degenerate problem. Otherwise, a modified solution should be applied (Wu, 1984). However, if the final solutions do not contain the eigenvectors \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \), the problems of repeated eigenvalues can then be avoided, which can usually be achieved through the use of the identities given in the next subsection.

As shown by Sano (1990a), the analyticity of a function is not affected by the arguments \( z_1, z_2, z_3, \) or \( z_4 \). Another solution form appropriate for the method of analytic continuation (Makkebelisnivili, 1954) is written as

\[ T = g(z) + \overline{g(\overline{z})}, \quad h_1 = -(k_1 + ik_2)g(z) - (k_1 + ik_2)g'(z), \quad (4a) \]

\[ \mathbf{u} = \mathbf{A}g(z) + \mathbf{c}g(z) + \mathbf{A}g(\overline{z}) + \mathbf{c}g(\overline{z}), \quad \mathbf{z} = \mathbf{B}g(z) + \mathbf{d}g(z) + \mathbf{B}g(\overline{z}) + \mathbf{d}g(\overline{z}), \quad (4b) \]

where

\[ \mathbf{A} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_1 & b_2 \\ b_1 & b_2 \end{bmatrix}, \quad \mathbf{f}(z) = \begin{bmatrix} f_1(z) & f_2(z) & f_3(z) \end{bmatrix}^T. \quad (4c) \]

and the overbar represents the conjugate of a complex number. Note that the arguments of \( g(z) \) and each component function of \( \mathbf{f}(z) \) are written as \( z = x_1 + ip_3 \) without referring to the associated eigenvalues \( \tau \) or \( \epsilon_3 \). Once the solutions of \( g(z) \) and \( \mathbf{f}(z) \) are obtained for a given boundary value problem, a replacement of \( z_1, z_2, z_3, \) or \( z_4 \) should be made for each function to calculate field quantities from (1) or (4).

### 2.2. Identities

Due to the orthogonality relation among the eigenvectors \( \mathbf{\xi} \), derived by Stroh (1958), three real matrices have been introduced as

\[ \mathbf{S} = i(2\mathbf{A}\mathbf{B}^T - \mathbf{I}), \quad \mathbf{H} = 2\mathbf{A}\mathbf{A}^T, \quad \mathbf{L} = -2\mathbf{B}\mathbf{B}^T. \quad (5) \]

in which \( i = \sqrt{-1} \) is a pure imaginary number and \( \mathbf{I} \) is the unit matrix. \( \mathbf{H} \) and \( \mathbf{L} \) are symmetric and positive definite and \( \mathbf{SH}, \mathbf{LS}, \mathbf{H}^{-1} \mathbf{S}, \mathbf{SL}^{-1} \) are anti-symmetric. From the
above relations, the impedance matrix \( M \) (Ingebritsen and Tonning, 1969) which has been used widely for the interface crack problems can be shown to be

\[
M = -iBA^{-1} - H^{-1}(I + S) = (I - iS^\top)H^{-1},
\]

\[
M^{-1} = iAB^{-1} - L^{-1}(I + S^\top) = (I - iS)L^{-1}.
\]

(6)

The third equalities in (6) come from the fact that \( H^{-1}S \) and \( SL^{-1} \) are anti-symmetric. Hence \( M \) is a Hermitian matrix. Another identity related to the thermoelastic properties is (Hwu, 1990)

\[
Sc + dS = i\epsilon + \frac{\partial}{\partial t}, \quad -LC + S^\top d = d + \frac{\partial}{\partial t},
\]

(7a)

where

\[
\gamma_1 = \frac{1}{2\pi} \int_{0}^{2\pi} (\cos \theta + \epsilon \sin \theta) y_1(\theta) \, d\theta, \quad y_1(\theta) = -N_i(\theta)b_m(\theta),
\]

\[
\gamma_2 = \frac{1}{2\pi} \int_{0}^{2\pi} (\cos \theta + \epsilon \sin \theta) y_2(\theta) \, d\theta, \quad y_2(\theta) = -\bm(m(\theta) - N_i^\top(\theta)b_m(\theta)),
\]

(7b)

and

\[
\mu(\theta) = (\cos \theta \sin \theta \theta)^{\top}, \quad m(\theta) = (-\sin \theta \cos \theta \theta)^{\top}, \quad \beta = [\beta_1, \beta_2, \beta_3].
\]

(7c)

\( N_i(\theta), i = 1, 2, 3 \) are the generalized forms of \( N_i \) (Hwu, 1990).

3. THERMOELASTIC COLLINEAR INTERFACE CRACKS

Consider an arbitrary number of collinear cracks lying along the interface of two dissimilar anisotropic materials. The materials are assumed to be perfectly bonded at all points of the interface \( x_0 = 0 \) except those lying in the region of cracks \( L \) (see Fig. 1), which are defined by the intervals

\[ a_1 < x_1 < b_1, \quad j = 1, 2, \ldots, n, \]

with

\[-\infty < a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < \infty.\]

On the upper and lower surfaces of the cracks, an arbitrary and self-equilibrated loading and heat flux are specified. From Section 2, we know that the solution to an individual

\[ \text{Fig. 1. Collinear interface cracks.} \]
problem in two-dimensional anisotropic thermoelasticity can be reduced to finding the complex functions $f$ and $g$, which should satisfy the boundary conditions of that problem. In the case of two different materials, however, the elastic properties are discontinuous across the bonded line, and a complete solution to the problem requires the knowledge of two complex function vectors $f_1$ and $f_2$ and two complex scalar functions $g_1$, $g_2$. Here, the subscripts 1 and 2 are used to denote the quantities pertaining to the materials 1 and 2 which are located on $x_1 > 0 \ (S_1)$ and $x_1 < 0 \ (S_2)$, respectively. The functions $f_1$, $g_1$, and $f_2$, $g_2$ are holomorphic in the regions $S_1$ and $S_2$, respectively. They are sought to satisfy the continuity of displacement, traction, temperature and heat flux across the bonded portion of the interface, as well as the prescribed traction and heat flux conditions on the crack portion, i.e.

$$
\begin{align*}
\mathbf{u}_1 &= \mathbf{u}_2, \\
\phi_1 &= \phi_2, \\
T_1 &= T_2, \\
\mathbf{h}_1 &= \mathbf{h}_2, \\
\phi_1 &= \phi_2, \\
(h_1)_1 &= (h_2)_2 = \tilde{h}_o, \\
x_1 &\in L.
\end{align*}
(8)
$$

The equality of traction continuity comes from the relation $\partial \mathbf{u}/\partial n = \mathbf{t}$ on the surface traction on a curve boundary and $s$ is the arc length measured along the curve boundary, $\mathbf{t}$ and $\tilde{h}$ are the prescribed traction and heat flux applied on the upper and lower surfaces of the cracks. When the points along the crack surfaces are considered, integration of $\phi_1 = \phi_2$ provides $\phi_1 = \phi_2$ since the integration constants can be neglected, which correspond to rigid body motion. Combining this result with the continuity requirements of traction and heat flux along the bonded portion, we have

$$
\phi_1 = \phi_2, \quad (h_1)_1 = (h_2)_2, \quad \text{along the entire interface.}
(9)
$$

By introducing a real constant $k = k_{ij}(t-f)/2l$, the expression of the heat flux component $h_j$ given in (4a), can be simplified as

$$
h_j = -ikg^j(z) + ik\overline{g^j}(\overline{z}).
(10)
$$

Using eqn (10), the heat flux continuity condition (9), leads to

$$
-ikg^j(x) + ik\overline{g^j}(\overline{x}) = -ikg^j(x) - ik\overline{g^j}(\overline{x}),
(11)
$$

where $x^j$ denote, respectively, the points on the upper and lower surfaces of the cracks. One of the important properties of holomorphic functions used in the method of analytical continuation is that if $f(z)$ is holomorphic in $S_1$ or $S_2$, then $\overline{f}(\overline{z})$ is holomorphic in $S_1$ or $S_2$. From this property and eqn (11), we may introduce a function which is holomorphic in the entire domain including the interface, i.e.

$$
g^*(z) = \begin{cases} 
-ikg^j(z) - ik\overline{g^j}(\overline{z}), & z \in S_1, \\
-ikg^j(z) - ik\overline{g^j}(\overline{z}), & z \in S_2.
\end{cases}
(12)
$$

Since $g^*(z)$ is now holomorphic and single-valued in the whole plate including the point at infinity, by Liouville's Theorem we have $g^*(z)$ = constant. If the heat flux tends to zero when $|z| \to \infty$, $g^*(z)$'s then identically zero, i.e.

$$
g^*(z) = 0.
(13)
$$

If the temperature field also tends to zero as $|z| \to \infty$, and the terms corresponding to the rigid body motion are neglected, combining (12) and (13) we have
\[
\bar{g}_2(z) = -\frac{k_2}{k_1} \bar{g}_1(z), \quad z \in S_1, \\
\bar{g}_1(z) = -\frac{k_1}{k_2} \bar{g}_2(z), \quad z \in S_2.
\] (14)

Similarly, by applying (4a)c, the traction continuity (9), along the entire interface and the result of (14), the use of the analytical continuation method leads to
\[
\bar{T}_2(z) = \bar{b}_2 \begin{bmatrix} B_{21} & \bar{b}_1 \\ B_{22} & \bar{b}_2 \end{bmatrix} \left[ \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \right] g_2(z), \quad z \in S_1, \\
\bar{T}_1(z) = \bar{b}_1 \begin{bmatrix} B_{11} & \bar{b}_1 \\ B_{12} & \bar{b}_2 \end{bmatrix} \left[ \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \right] g_2(z), \quad z \in S_2.
\] (15)

By employing (4a)c, (14) and (15), the temperature and displacement continuity along the bonded portion of the interface now provide
\[
\theta(x') = \theta(x), \quad \psi(x') = \psi(x), \quad x \in \mathcal{L},
\] (16a)

where
\[
\theta(z) = \begin{cases} 
1 + \frac{k_1}{k_2} \bar{g}_1(z), & z \in S_1, \\
1 + \frac{k_2}{k_1} \bar{g}_2(z), & z \in S_2.
\end{cases}
\] (16b)

\[
\psi(z) = \begin{cases} 
B_{21} d_1 + \frac{k_2}{k_1+k_2} d_2 + e_1, & z \in S_1, \\
M^* \bar{M}^* B_{21} d_1 + \frac{k_2}{k_1+k_2} d_2 + e_1, & z \in S_2.
\end{cases}
\] (16c)

In the above, \(M^*\) is the bimaterial matrix defined as
\[
M^* = M_1^{-1} - \bar{M}_1^{-1} = (A_1 B_1^{-1} - \bar{A}_1 \bar{B}_1^{-1}) = -(W + iD),
\] (17a)

where
\[
W = S_1 L_1^{-1} - S_2 L_2^{-1}, \quad D = L_1^{-1} - L_2^{-1}.
\] (17b)

The second and third equalities of (17a) come from the identities given in (6). The complex vectors \(e_1\) and \(e_2\) are related to the heat eigenvectors \(e\) and \(d\), and are defined as
\[
e_1 = \frac{1}{k_1+k_2} [M^* e - i\bar{M}_1^{-1} d] - \frac{k_2}{k_1+k_2} d_2,
\]
\[
e_2 = \frac{1}{k_1+k_2} [\bar{M}^* e - i\bar{M}_1^{-1} d] - \frac{\bar{k}_2}{k_1+k_2} d_2.
\] (18a)

where
\[
e = k_1 e_1 + k_2 e_2, \quad d = k_1 d_1 + k_2 d_2.
\] (18b)

Using the results of (14), (15) and (16b,c), the prescribed traction and heat flux conditions on the crack portion (8b)c lead to the following Hilbert problems (Muskheleshvili, 1954).
\[ \psi'(x_1') + \Phi'(x_2') = \frac{\Gamma_1 + k_1}{k_2 k_3} \phi(x_1'), \quad x_1' \in L. \]  

The solutions to these Hilbert problems are (see Appendix)

\[
\psi'(x) = \frac{k_1 + k_3}{2k_2} x_2 X_0(\theta) \int \frac{h_0(x)}{k_2} (x_2 - \theta) \, d\theta + \chi_0(x) \phi(x) \tag{20a}
\]

\[
\psi'(x) = \frac{1}{2\pi} x_2 X_0(\theta) \int \frac{-1}{x_2 - \theta} (X_2^1(\theta) + \Phi'(x_3') x_3' + \Phi'(x_3') x_3') \, dx + X_2(\theta) \phi(x) \tag{20b}
\]

where \( \phi(x) \) and \( \phi(x) \) are arbitrary polynomials with the degree not higher than \( x \) and \( x_0 \), \( X_0(x) \) are the basic Plemelj functions defined as,

\[ \chi_0(x) = \prod_{i=1}^{n} (x - a_i)^{-\frac{1}{2}}(x - b_i)^{-\frac{1}{2}}, \quad X_0(x) = \Lambda \Gamma(x), \tag{26c} \]

where

\[ \Lambda = [k_1, k_2, k_3], \quad \Gamma(x) = \left\{ \prod_{i=1}^{n} (x - a_i)^{-\frac{1}{2}}(x - b_i)^{-\frac{1}{2}} \right\}. \tag{20d} \]

The angular bracket \( \langle \langle \rangle \rangle \) stands for the diagonal matrix, i.e.,

\[ \langle \langle f(x) \rangle \rangle = \text{diag} \{ f_1, f_2, f_3 \} \]

which will be used throughout this paper. \( \delta_1 \) and \( \lambda_0 \), \( x = 1, 2, 3, \) of (20d) are the eigenvalues and eigenvectors of

\[ \left( M^2 + \epsilon \beta_{\text{eff}} \Phi^* \right) \lambda = 0. \tag{26d} \]

The explicit solution for the eigenvalue \( \delta_1 \) has been given by Ting (1986) as

\[ \delta_1 = -\frac{1}{2} + i\xi, \quad \xi = 1, 2, 3, \tag{20e} \]

where

\[ \epsilon_1 = \delta = \frac{1}{2} \ln \left| \frac{1 + \epsilon_0}{1 - \epsilon_0} \right|, \quad \epsilon_2 = -\delta, \quad \epsilon_3 = 0, \quad \omega = -\frac{1}{2} \text{tr} \left( W D^{-1} D^{-1} \right)^{1/2}, \tag{26g} \]

which stands for the trace of the matrix. Note that the order of singularity \( \delta_1 \) is independent of the heat conduction coefficients \( k_1 \) and \( k_2 \), and the thermal moduli \( k_3 \) and \( k_4 \) is the same as those of the isothermal interface crack problem. The order of singularity related to the heat flux is \( -1/2 \) as shown in (26c).

Once we get the solution of \( \Phi'(x) \) and \( \psi'(x) \) from (20), the complex functions \( g_1(x) \), \( g_2(x) \), \( f_1(x) \), \( f_2(x) \), \( f_3(x) \) can be obtained from (16b, c) with the understanding that the subscript of \( z \) is dropped since the analytical continuation is not affected by different arguments \( z_1 \) or \( z_2 \). After the operation of matrices, a replacement of \( z_2 \) or \( z_1 \), \( z_2 \), \( z_3 \) should be made for each function, because the functions \( g_k(z) \) and \( f_k(z) \) are required to have the form

\[ g_k(z) = g_0(z), \quad f_k(z) = \left[ f_1(z_1) f_2(z_1) f_3(z_1) \right], \quad k = 1, 2, \]

which can be seen from the general solution given in (1). This calculation procedure will be applied throughout this paper. The whole field solution can then be found by using eqn (4). If \( \epsilon \) is interested in the stresses \( \sigma_{ij} \) along the interface and the crack opening
displacements \( \Delta u \), the following results show that they have a simple relation with functions \( \phi(z) \) and \( \theta(z) \). By applying \( \sigma_{ij} = \phi_{,ij} \), (6a), (14), (15) and (16b,c), the stresses \( \sigma_{ij} \) are calculated as

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12}
\end{bmatrix} = \phi^{'} = (1+\mathbf{M} \mathbf{k} \mathbf{M}^{T})\psi(x_{1}) - \theta(x_{1})(e_{1} + e_{2}), \quad x_{1} \notin L.
\]

(21)

From (4a), (14), (15) and (16b,c), the crack opening displacements \( \Delta u \) can also be calculated and simplified as

\[
\Delta u = u(x_{1},0^{+}) - u(x_{1},0^{-}) = -\mathbf{M}^{T}\phi(x_{1}) - \phi(x_{1}), \quad x_{1} \in L.
\]

(22)

4. EXAMPLES

4.1. Homogeneous media

The simplest case of the interface cracks is when the two media are composed of the same materials. Our results for the thermoelastic interface crack problems should therefore be checked by this simplest case. Consider an infinite homogeneous anisotropic plate containing an insulated crack in which the heat is flowing uniformly in the direction of the positive \( x_{2} \)-axis. Due to the linear property, the principle of superposition can be used and the problem can be represented as the sum of a uniform heat flux in an uncracked solid and corrective problem which is described by

\[
\begin{align*}
\bar{h}(x_{1}) &= -h_{x} = \text{constant}, \quad \bar{h}(x_{1}) = 0, \quad n = 1, \quad a_{i} = -a, \quad b_{i} = a, \\
A_{i} &= A_{j} = A, \quad B_{i} = B_{j} = B, \quad c_{1} = c_{2} = c, \quad d_{1} = d_{2} = d, \quad k_{1} = k_{2} = k.
\end{align*}
\]

(23)

To find the solution for this corrective problem, the line integral given in (20a) should be evaluated first. By residue theory, the integral around a closed contour \( C \) shown in Fig. 2 can be calculated as

\[
\oint_{C} \frac{h_{x} \mu}{\kappa_{j}(\sigma_{ij})} dr = 2\pi i \sum \notag_{\gamma} r_{i} Z(\gamma) = \frac{1}{\sqrt{\gamma_{f} - \gamma_{s}}},
\]

where \( r_{i} \) is the residue of the integrand at its singular points within \( C \). The closed contour \( C \) is the union of \( L_{r} \), \( C_{r} \), \( L_{r} \), \( L_{r} \), \( C_{r} \), \( L_{r} \). The summation of the integrals along \( L_{r} \) and \( L_{l} \) vanishes since they have opposite directions and the integrand across this line is continuous. The integrals around the circles \( C_{r} \) and \( C_{l} \) can be proved to be zero when the

Fig. 2. Integration contour.
 naïve of the circles \( C_0 \) and \( C_1 \) tend to zero. By replacing the contour of \( C_0 \) by \( \Re e^w \) and letting \( R \to \infty \), the integral around \( C_0 \) is found to be
\[
\int_{\gamma_0} \frac{h_0}{z_0}(v - z)\,dz = \alpha h_0 \log z.
\]

Knowing that \( z_0(z) \) is the homogeneous solution of the Hilbert problem, i.e. \( z_0^2 + \omega^2 = 0 \), we have
\[
\int_{\gamma_0 + \epsilon} \frac{h_0}{z_0} \,dz = \int_{\gamma_0} \frac{h_0}{z_0} \,dz + \int_{\gamma_0} \frac{h_0}{z_0} \,dz.
\]
The only pole which has made a contribution to the residues is at \( s = z_0 \), and the residue at that point is \( h_0 \gamma_0^2(z) \). With the above description, we are now in a position to evaluate the line integral and the final simplified result is
\[
\int_{\gamma_0} \frac{h_0}{z_0} \,dz = -\pi h_0 [z - \sqrt{z^2 - a^2}]. \tag{24a}
\]

After evaluating the line integral, the arbitrary polynomial
\[
p(z) = c_0 + c_1 z
\]
could be determined by the infinity condition and the single-valuedness requirement. If \( \theta'(z) \to 0 \) as \( |z| \to \infty \), we have
\[
c_1 = 0. \tag{24b}
\]
The requirement of the single-valuedness condition can be expressed by
\[
\int_{\gamma_0} [\theta''(z_i) - \theta''(z_i)] \,dz_i = 0. \tag{24c}
\]
Knowing that \( \sqrt{z^2 - a^2} = \pm i \sqrt{z^2 - x_0^2} \) for \( |x_i| < a \) and \( x_0 = \pm 0 \), substitution of (24a) and (20a) into (24d) leads to
\[
c_0 = 0. \tag{24d}
\]
Combining eqs (24a)–(e), the final simplified result for \( \theta''(z) \) is
\[
\theta''(z) = -\frac{h_0}{k} \left( 1 - \frac{z}{\sqrt{z^2 - a^2}} \right). \tag{25}
\]

To find the solution for \( \phi(z) \), we first calculate the terms related to \( \theta'(z) \). Integrating \( \theta'(z) \) given in (25) with the assumption that the temperature field tends to zero as \( |z| \to \infty \), and substituting the identities (6) and (7a) into (18) with the homogeneous condition (23), we have
\[
\theta'(z) \psi + \theta'(z) \phi = -\frac{h_0}{k} \Re e \{ \phi \}. \tag{26}
\]

The evaluation of the line integral and the determination of the arbitrary polynomial are similar to those described in eqs (24a)–(e). The result is
The solutions of the complex functions \( g(z) \) and \( f(z) \) are obtained from (16b,c) with the aid of (23)_{1,4} and (7a)_{2} as

\[
\begin{align*}
g(z) &= \frac{\Theta(z)}{\ell} \\
f(z) &= [B^{-1}] \left( 2\Phi(z) = \Theta(z) \mathbf{d} - i \text{Re} \{\Theta(z)\} \right).
\end{align*}
\]  

Note again that the subscript of \( s \) is dropped before the multiplication of matrices and a replacement of \( z_{1}, z_{2} \) should be made for each component function after the matrix product.

By this calculation procedure, the explicit expressions for the complex functions \( g(z) \) and \( f(z) \) can be obtained from (23)--(27) as

\[
\begin{align*}
g(z) &= -\frac{i\hbar_{s}}{4k} \left( z_{1}^{2} - z_{2} \sqrt{z_{1}^{2} - a^{2}} + a^{2} \log \left( z_{1} + \sqrt{z_{1}^{2} - a^{2}} \right) \right), \\
f(z) &= \frac{i\hbar_{s}}{4k} \left( \left\langle z_{1}^{2} - z_{2} \sqrt{z_{1}^{2} - a^{2}} \right\rangle \right) B^{-1} \mathbf{d} \\
&\quad + \frac{i\hbar_{s}a^{3}}{4k} \left( \left\langle \log \left( z_{1} + \sqrt{z_{1}^{2} - a^{2}} \right) \right\rangle \right) B^{-1} \mathbf{d} - i \text{Re} \{\Theta(z)\}.
\end{align*}
\]  

which can be proved to be identical to those presented in Hwu (1990) through the use of identities given in (5) and (7a). The whole field solutions for the temperature, heat flux, displacements and stresses can then be found by using eqn (4). The stresses \( \sigma_{22} \) ahead of the crack tip along the \( x_{1} \)-axis are calculated by (21) with \( \theta \) and \( \psi \) given in (25) and (26) as

\[
\sigma_{22} = \frac{h_{s}}{2k} \frac{a^{2}}{\sqrt{x_{1}^{2} - a^{2}}} \text{Re} \{\Theta(z)\}.  \tag{29a}
\]

Same as the isothermal problems, the above solution shows that the stresses are singular near the crack tip. With the usual definition, the stress intensity factors are given by

\[
\begin{bmatrix}
K_{I} \\
K_{II}
\end{bmatrix} = \lim_{\sigma_{22} \to \infty} \sqrt{2\pi(x_{1} - a)} \sigma_{22} = \frac{\sqrt{\hbar_{s}a^{3}}}{2k} \text{Re} \{\Theta(z)\}.  \tag{29b}
\]

Similarly, the crack opening displacements \( \Delta u \) are obtained from (22) and (26) as

\[
\Delta u = u(x_{1}, 0^{+}) - u(x_{1}, 0^{-}) = \frac{h_{s}}{4} x_{1} \sqrt{x_{1}^{2} - a^{2}} L^{-1} \text{Re} \{\Theta(z)\}.  \tag{29c}
\]

The validity of this solution related to the assumption of a fully open crack has been discussed by Hwu (1990). By applying the virtual crack closure method (Irwin, 1957), the total strain energy release rate \( G \) can then be calculated as

\[
G = \lim_{\Delta u \to 0} \frac{1}{2\Delta u} \int_{0}^{\pi} u(x - \Delta u) \sigma_{22}(x) \, dx = \frac{\delta^{2} \hbar_{s} a^{3}}{8k} \text{Re} \{\Theta(z)\} L^{-1} \text{Re} \{\Theta(z)\}.  \tag{29d}
\]

The solutions given in eqns (29a-d) are also exactly the same as those presented in Hwu (1990).
4.2. Biomaterials

Consider an interface crack located on \( a_1 = -a_2, b_1 = a_2 \), subjected to uniform hoop flux \( h = -b_2 \) and uniform loading \( f = -t_0 \). To find the solution of \( \phi'(z) \) from (20a), a similar approach to that in Section 4.1 can be employed and the result is

\[
\phi'(z) = -ih^2 \left( z - \sqrt{z^2 - a^2} \right),
\]

where

\[
h^2 = \frac{h_0 (k_1 + k_2)}{2k_1 k_2}.
\]

With the aid of residue theory, the following line integrals which are useful for the calculation of \( \psi'(z) \) can be obtained in a similar manner to those described in Section 4.1:

\[
\frac{1}{2\pi i} \int_{s-z} \frac{[X_2(\zeta)]^{-1} \zeta_0}{s-\zeta} \mathrm{d}\zeta = \left< \left( \frac{1}{X_2(z)} - \frac{z + 2i\alpha a}{2} \right) \right> \psi^*,
\]

\[
\frac{1}{2\pi i} \int_{s-z} \frac{s - \zeta}{X_2(\zeta)^{-1} \zeta_0} \mathrm{d}\zeta = \left< \left( \frac{z + 2i\alpha a}{X_2(z)} - \frac{a^2}{2(1 + 4i\alpha^2)} \right) \right> \psi^*,
\]

\[
\frac{1}{2\pi i} \int_{s-z} \frac{\sqrt{s^2 - a^2} - \zeta}{X_2(\zeta)^{-1} \zeta_0} \mathrm{d}\zeta = \left< \left( \frac{s^2 - a^2 - 2i(z + 2i\alpha a) - a^2(1 + 2i\alpha)}{X_2(z)} \right) \right> \psi^*.
\]

(31a)

where

\[
\psi^* = A^{-1} \left( 1 + \bar{M}^* \bar{M}^* \right)^{-1} t_0, \quad \psi^* = A^{-1} \left( 1 + \bar{M}^* \bar{M}^* \right)^{-1} \zeta_0, \quad k = 1, 2,
\]

\[
X_2(z) = \frac{1}{\sqrt{s^2 - a^2}} \left( \frac{z - a}{s + a} \right).
\]

(31b)

Applying the results of (30) and (31), the arbitrary polynomial \( p_z(z) \) in (20b) can now be determined by the infinity condition,

\[
\psi'(z) \to 0, \quad \text{as} \quad |z| \to \infty,
\]

and the single-valuedness requirement,

\[
\int_{-\infty}^{\infty} \left[ \psi'(x) - \psi'(x^*) \right] \mathrm{d}x = 0.
\]

The result is

\[
p_z(z) = -ih^2 \left\{ \frac{a^2}{4} \psi^* + \left< 4ia \alpha a^2 - a^2(1 + 4i\alpha^2) \right> \psi^* \right\}.
\]

(32)

In the derivation of (32) the following integrals calculated by the residue theory have been used.
Combining the results of (30)-(32), the final simplified solution for $\psi(\zeta)$ is found to be

$$\psi(\zeta) = -A \left[ J_0(\zeta) + iM(\zeta) \right],$$

(33a)

where

$$J_0(\zeta) = \langle \chi - (z + 2i\omega_0) \phi(\zeta) \rangle,$$

$$J_1(\zeta) = \langle \chi - (z - \omega_0) \phi(\zeta) \rangle,$$

$$J_2(\zeta) = \langle \chi + (z - \omega_0 - (2z^2 - \omega_0^2) \phi(\zeta) \rangle.$$  

(33b)

The solutions for the isothermal interface crack problems are found by letting $\delta = 0$, which can be proved to be identical to those presented in Wu (1990) and Suo (1990a). The fracture parameters such as the stress intensity factors, crack-opening displacements and the energy release rate can then be obtained from this solution and the proper definition for the interface crack problems (Wu, 1990; Suo, 1990a; Hwu, 1993).

5. CONCLUSIONS

A general solution for the thermoelastic collinear interface cracks between dissimilar anisotropic media has been obtained by applying the extended version of Stroh's formalism and a special technique of analytical continuation. The general solution is valid when the heat eigenvalue and the elasticity eigenvalues are distinct. For the case that they are repeated, a small perturbation of the material constants can be employed to avoid the degenerate problem. Otherwise, a modified solution should be developed. For the solutions in which the eigenvectors have been replaced by the fundamental matrices such as $S$, $H$, $\Omega$, $\delta$ through the use of identities, the problems of repeated eigenvalues disappear, which has been demonstrated in the case of homogenous media.

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REFERENCES


APPENDIX: SOLUTIONS TO THE DILBERT PROBLEM OF VECTOR FORM

The Hilbert problem is usually expressed in the form of scalar functions,

\[ F(x) \rightarrow \phi(x) = f(x) \quad \text{on } L, \quad \text{except at the ends.} \]

where \( \phi \) is a general real or complex constant. The solution to this problem with \( \phi \neq 0 \) has been shown in M. Williams (1959).

\[ F(x) = \sum_{k=1}^{\infty} \frac{f(i) \delta_{i,j}}{\sqrt{i}} + \sum_{k=1}^{\infty} \alpha_{i,j} \text{polynomial} \]

where \( \alpha_{i,j} = \oint_{L} \frac{(x - x)}{\pi} \text{polynomial} \).

\( y = (1/\sum_{k} \ln |y|) \) is an arbitrary polynomial with the degree not higher than \( n \). For \( y = 1 \), the solution is

\[ f(x) = \sum_{k=1}^{\infty} \frac{f(i) \delta_{i,j}}{\sqrt{i}} + \sum_{k=1}^{\infty} \alpha_{i,j} \text{polynomial} \]

To find the solution for the vector form expression,

\[ \psi(x) + \int_{0}^{x} \psi(x') dx' = \mathbf{L}, \quad x \in L, \quad \text{except at the ends}, \]

similar approach can be employed. Firstly, a solution will be sought which may have a pole of arbitrary order at infinity, and we begin with the homogeneous problem,

\[ \psi(x) + \int_{0}^{x} \psi(x') dx' = 0, \quad x \in L \]

A particular solution \( \psi(x) \) of the problem will be sought in the form

\[ \psi(x) = \sum_{k=1}^{\infty} (x - x_k)^{-\alpha} \text{polynomial} \]

where \( \alpha \) is a complex constant and \( L \) is a complex constant vector. The function \( \psi(x) \) is holomorphic in the entire plane cut along \( L, \) if a different branch of this function is assumed. It is readily verified by an investigation of the variation in the argument of \( x \to a, a \to b, \) when \( x \) describes a closed path beginning at point \( a \) of the arc \( a, b, \) and ending, without intersecting \( L, \) from the left side of \( a, b, \) to the right side of the arc or around the end \( b, \) then

\[ \psi(x) = e^{\alpha} \psi(x) \]

Hence, \( \psi(x) \) will satisfy the boundary condition (3.5), provided.
\[ (e^{i\theta} - 5)^{-1} Y^* = 0. \]

or

\[ (\mathbf{M}^* + e^{i\theta} \mathbf{M}) \mathbf{l} = 0. \] (A8)

The explicit solution for the eigenvalue \( \lambda \) has been given by Ting (1986) as

\[ \lambda_1 = -1 + i, \quad \lambda_2 = -1 - i, \quad \lambda_3 = -1, \] (A9)

where

\[ \tau = \frac{1}{2\pi} \ln \left( \frac{1 + \nu}{1 - \nu} \right), \quad \omega = - \left[ \tau \mathbf{W}^T \mathbf{W} \right]^{1/3}, \]

\( \omega \) stands for the trace of the matrix. Thus, a particular solution \( \psi(x) \) of the homogeneous problem has been found; it is given by (A8) with \( \omega = \lambda \) determined by (A8). Since there are three eigenvalues from (A8), a linear combination of these particular solutions will be one of the particular solutions, i.e.,

\[ \psi(x) = X(x)p_x, \] (A10)

where

\[ X(x) = \mathbf{A}^T(x), \] (A11)

and

\[ \mathbf{A} = [\mathbf{A}_1 \quad \mathbf{A}_2 \quad \mathbf{A}_3], \quad \mathbf{F}(x) = \left( \begin{array}{c} 1 \\ (x - \omega)^{1/3} \\ (x - \omega)^{-1/3} \end{array} \right). \] (A12)

\( p_x \) is a coefficient vector. This particular solution does not vanish anywhere in the finite part of the plane and it is unbounded like \( (x - \omega)^{1/3} \) and \( (x - \omega)^{-1/3} \) near the ends \( \omega_1 \) and \( \omega_2 \), respectively.

The most general solution of the homogeneous problem will now be found which has a pole at infinity. For the purpose it will be noted that \( \psi(x) = X(x)p_x \), being a solution of the homogeneous problem, satisfies the condition

\[ X(x)p_x + \mathbf{S}^* \mathbf{M}^* X(x)p_x = 0, \quad x \notin L. \]

Hence

\[ \mathbf{S}^* \mathbf{M}^* = -X(x)^T \mathbf{X}(x)^{-1}, \] (A13)

where \( \mathbf{X}(x) \) is the simplified notation for \( \mathbf{X}(x)^T \). By applying (A13), eqn (A5) becomes

\[ \mathbf{X}(x)^T \mathbf{\Psi}(x) - \mathbf{X}(x)^T \mathbf{\Psi}(x) = 0, \quad x \in L. \]

or

\[ \mathbf{\Psi}(x) - \mathbf{\Psi}(x) = 0, \quad x \in L, \] (A14)

where \( \mathbf{\Psi}(x) \) denotes the sectorially holomorphic function \( \mathbf{X}(x)^T \mathbf{\Psi}(x) \). It follows from (A14) that \( \mathbf{\Psi}(x) \) is holomorphic in the entire plane, except at the point \( x = \omega \), provided it is given suitable values on \( L \). Further, since \( \mathbf{\Psi}(x) \) can only have a pole at infinity, it must, by the generalized Liouville theorem, be a polynomial. Thus, the most general solution of the homogeneous problem is given by

\[ \mathbf{\Psi}(x) = \mathbf{X}(x)p_x(x), \] (A15)

where \( p_x(x) \) is an arbitrary polynomial vector. If it is desired to obtain a solution which is also holomorphic at infinity, it must be assumed that the degree of the polynomial \( p_x(x) \) does not exceed \( n \). This follows from the behavior of \( \mathbf{X}(x) \) at infinity as given in (A11).

Next consider the non-homogeneous problem. Using (A13), the boundary condition (A4) may be written as

\[ \mathbf{X}(x)^T \mathbf{\Psi}(x) = \mathbf{X}(x)^T \mathbf{\Psi}(x) = \mathbf{X}(x)^T \mathbf{L}, \quad x \in L, \]

or

\[ \mathbf{\Psi}(x) - \mathbf{\Psi}(x) = \mathbf{X}(x)^T \mathbf{L}, \quad x \in L, \] (A16)

where \( \mathbf{\Psi}(x) = \mathbf{X}(x)^T \mathbf{\Psi}(x) \). Each component of eqn (A16) is in the form of (A1) with \( \nu = 1 \), hence, by (A3) we have

\[ \mathbf{\Psi}(x) = \frac{1}{2\pi i} \mathbf{X}(x) \int_{\gamma} \frac{1}{\mathbf{X}(x)^T \mathbf{\Psi}(x) - \mathbf{X}(x)^T \mathbf{\Psi}(x)} \mathbf{d}x + \mathbf{X}(x)p_x(x) \] (A17)

where \( p_x(x) \) is an arbitrary polynomial vector with the degree not higher than \( n \).