Three-dimensional Analysis of Generally Anisotropic Piezoelectric Materials by the BEM Based upon Radon-Stroh Formalism

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Abstract. This paper presents a piezoelectric study for three-dimensional (3D) anisotropic structures by the boundary element method (BEM) with fundamental solutions formulated using real variable Radon-Stroh formalism. As in 2D cases, pertinent equations of the 3D cases are formulated by simply combining matrices of elasticity as well as those of piezoelectricity into extended forms. Also, evaluations of the fundamental solutions by the Radon-Stroh formalism are discussed. In the end, an illustrative example is presented.

Introduction

In advanced structures, piezoelectric materials have been widely applied for various purposes. A great amount of literatures of BEM have been reported in this regard, they are mostly for two-dimensional (2D) problems though. For successful implementation of BEM to practical applications, the key step is to evaluate the fundamental solutions in the boundary integral equation. Due to mathematical complexity involved, evaluations of the fundamental solutions of 3D anisotropic elasticity have been a focus in the BEM community (e.g [1]-[5]); however, studies on its derivatives have remained very scarce indeed. Among these, the approach by Radon transform [2,6] based on the Stroh formalism has revealed itself usefulness when applied to construct complicated 3D Green’s functions from corresponding 2D solutions to particular problems, such as those of half-space [2] and bi-materials [7]. The well-known Stroh formalism can be used to obtain solutions of the particular 2D problems, by which the corresponding 3D solutions can thus be constructed simply by integrating those of the 2D cases. For saving computation costs in calculating Stroh’s eigenvalues and eigenvectors for the complex solutions, real variable solutions can be formulated using 2D Stroh’s identities as presented in [8,9].

It is well known that the classical solutions for piezoelectric effects can be obtained by simply combining all related matrices of elasticity as well as piezoelectricity into the same mathematical forms but with expanded dimensions in the Stroh formalism. Among the very scarce works to treat 3D problems, Xie et al. [10,11] presented this extended approach for piezoelectric and magnetoelastic materials, where the Green’s function was obtained by both residue calculus and Stroh eigenvector method but not the Radon-Stroh formalism; however, no further implementation has been reported. Previously, the present authors had demonstrated a BEM analysis for pure elastic analysis based upon real variable Radon-Stroh formalism [9]. As in 2D cases, the 3D fundamental solutions for piezoelectric solids can also be obtained using the Radon-Stroh formalism and the expansion of matrices. In this paper, this methodology is implemented in BEM to study the piezoelectric effects in 3D generally anisotropic materials, where the Green’s function is derived based upon the real variable Radon-Stroh formalism. For verification, analyses by the commercial software ANSYS were also performed to make comparison with our BEM results.

Extended Stroh formalism

Before presenting the extended Stroh formalism for piezoelectric anisotropic elasticity, some basic equations are reviewed here, which include the constitutive law of piezoelectric solids, strain-displacement relationship, equilibrium equation and electrostatic equation, given as follows [12]:

\[
\begin{align*}
\sigma_{ij} &= C_{ijkl} e_{kl} - e_{ijkl} E_k, \\
D_j &= e_{ijkl} e_{ij} - \omega_{jk} E_k, \\
e_{ij} &= \frac{1}{2}(u_{ij} + u_{ji}), \\
\sigma_{i,j} &= 0, \\
D_{ij} &= 0, \\
&i, j, k, l = 1, 2, 3,
\end{align*}
\]
where \( u_i \) is the displacement component in \( x_i \)-axis, \( \sigma_{ij} \) and \( \epsilon_{ij} \) are respectively the stress and strain tensors, \( C_{ijkl} \) is the elastic stiffness tensor, \( e_{ijkl} \) and \( \omega_{ijkl} \) are piezoelectric constants and dielectric permittivity, \( D_i \) and \( E_i \) are electric displacements and electric field, respectively. In eq (1), by following tensor notations, the repeated indices imply summation and a comma stands for differentiation. By letting

\[
\begin{align*}
D_j &= \sigma_{ij}, \quad -E_j = u_{kj} = 2\epsilon_{kj}, \\
C_{ijkl} &= C_{ijkl}^e, \\
C_{ijkl} &= C_{ijkl}^{ijkl}, \\
C_{ijkl} &= \omega_{ijkl},
\end{align*}
\]

the basic equations (1) can be rewritten in an expanded notation as

\[
\sigma_{ij} = C_{ijkl}^{ijkl} \epsilon_{kl}, \quad \epsilon_{ij} = \frac{1}{2}(u_{ij,j} + u_{ij,j}), \quad \sigma_{ij,j} = 0, \quad I, J, K, L = 1, 2, 3, 4,
\]

in which the assumptions of \( u_{j,4} = 0, C_{ijkl}^{ijkl} = C_{ijkl}^{ijkl}, C_{ijkl}^{ijkl} = C_{ijkl}^{ijkl}, \sigma_{ij} = \sigma_{ij}, \) and \( \epsilon_{ij} = \epsilon_{ij} \) have been made, and the components of \( \sigma_{ij}, \epsilon_{ij}, C_{ijkl}, \) and \( C_{ijkl} \) are undefined. To avoid the undefined symbols, sometimes eq (3) was written as

\[
\sigma_{ij} = C_{ijkl}^{ijkl} \epsilon_{kl}, \quad \epsilon_{ij} = \frac{1}{2}(u_{ij,j} + u_{ij,j}), \quad \sigma_{ij,j} = 0, \quad I, K = 1, 2, 3, 4, \quad j, l = 1, 2, 3.
\]

It should be careful that if eq (4) is considered, the relation \(-E_j = u_{kj} = 2\epsilon_{kj}\) should be changed to \(-E_j = u_{kj} = \epsilon_{kj}\), and the assumption \( u_{j,4} = 0 \) should also be changed to \( u_{j,4} = \epsilon_{j,4} \). To be consistent with all the basic equations used for the anisotropic elastic materials, in this study we will follow the expressions given in eq (2) and (3) instead of eq (4).

**Boundary integral equation for 3D anisotropic piezoelectric analysis**

With the foregoing illustrations for dimension expansions, the boundary integral equation (BIE) for 3D electroelastic analysis can now be written in the same form as the conventional one for elastostatics [14,15], namely

\[
c_{ij}(\hat{\mathbf{x}}) u_{ij}(\hat{\mathbf{x}}) + \int_{\Gamma} t^*_{ij}(\hat{\mathbf{x}}, \mathbf{x}) u_{ij}(\mathbf{x}) d\Gamma(\mathbf{x}) = \int_{\Gamma} u^*_{ij}(\hat{\mathbf{x}}, \mathbf{x}) t_{ij}(\mathbf{x}) d\Gamma(\mathbf{x}),
\]

where all notations follow the same definitions for elastostatic analysis except \( u_4 \), being the electric potential, and \( t_4 \), being the electric displacement along the normal direction \( \mathbf{v} = (v_1, v_2, v_3) \), i.e.,

\[
t_4 = \sigma_{4j} v_j = D_j v_j = D_v.
\]

In eq (5), \( u^*_{ij}(\hat{\mathbf{x}}, \mathbf{x}) \) and \( t^*_{ij}(\hat{\mathbf{x}}, \mathbf{x}) \) are the generalized fundamental solutions of displacements and tractions in \( x_j \)-direction at the field point \( \mathbf{x} \) with point force/charge acting in the \( x_i \)-direction applied at point \( \hat{\mathbf{x}} \); \( c_{ij}(\hat{\mathbf{x}}) \) are the free term coefficients of the source point \( \hat{\mathbf{x}} \), which can be determined from the relation of rigid body motion as usual.

By using the expanded matrix form, the mathematical expression of the fundamental solutions for 3D electroelastic analysis remain the same as those for 3D anisotropic elasticity [7,13,14]. With reference to [8,9], the solutions formulated using the real variable Radon-Stroh formalism can be written in the following matrix forms:
where

\[
\hat{N}_i(x, y) = -\sin\psi N_i(x, y), \quad \hat{M}_i(x, y) = \cos\psi M_i(x, y) + \sin\psi M'_i(x, y),
\]

\[
M'_i(x, y) = [-\sin^2\psi N_i(x, y)]' + \cos2\psi I,
\]

\[
N'_i(x, y) = N_i(\theta)N_i(x, y) + N'_i(y)N_i(x, y).
\]

In eq (7a), \(\text{sgn}(\Delta x_i) = 1\) for \(\Delta x_i > 0\), \(\text{sgn}(\Delta x_i) = -1\) for \(\Delta x_i < 0\), and \(\Delta x_i = x_i - \hat{x}_i, i = 1, 2, 3\); \((x_1, x_2, x_3)\) and \((\hat{x}_1, \hat{x}_2, \hat{x}_3)\) are, respectively, the locations of field point \(x\) and source point \(\hat{x}\); \(N_i(\theta)\) and \(N_i(x, y)\), \(i = 1, 2, 3\) are the Stroh's fundamental elasticity matrices on the 2D Radon plane whose normal is related to the angle \(\theta\), \(r\) and \(\psi\) are the distance and polar angle related to the coordinates of Radon plane by

\[
r = \sqrt{\rho^2 + (\Delta x_i)^2}, \quad \psi = \tan^{-1}\frac{\Delta x_i}{\rho},
\]

\[
\rho = \Delta x_i \cos\theta + \Delta x_2 \sin\theta.
\]

\[
\cos\psi = \frac{\partial \rho}{\partial s}\text{ and } \sin\psi = \frac{\partial \rho}{\partial s}\text{ where } s\text{ is a parameter denoting the tangential direction of the boundary surface, and } \rho\text{ and } x_3\text{ are the coordinates used in 2D Radon-domain. The superscript } T\text{ represents the transpose of a matrix; the prime } '\text{ denotes the derivative with respect to } \psi, \text{ and the derivative of the generalized fundamental matrices } N_i(x, y)\text{ can be calculated by using the identities obtained for 2D anisotropic elasticity [9, 14].}
\]

With the free term coefficient set to be unity, the corresponding BIE for calculating displacement gradients at internal points can be obtained by directly differentiating eq (5) with respect to \(\hat{x}_K\), yielding

\[
\int_{\Gamma} u'_{j, K}(\hat{x}, \hat{\mathbf{x}}) u_j(\mathbf{x}) d\Gamma(\hat{x}) = \int_{\Gamma} u'_{j, K}(\hat{x}, \hat{\mathbf{x}}) u_j(\mathbf{x}) d\Gamma(\mathbf{x}).
\]

\[
\text{In eq (9), the derivatives of fundamental solutions, } [u'_{j, K}]_x \text{ and } [u'_{j, K}]_y, \text{ can be obtained directly from eq (7).}
\]

**Numerical examples**

All the presented formulations have been implemented in an existing BEM code. To demonstrate our successful implementation, an analysis was performed for a PZT cube excavated with a cylindrical through hole when uniform pressure is applied on top, and the bottom is totally fixed (Fig. 1(a)). Also, internal analyses were conducted for giving solutions at points along \(r = R_2\). For treating general anisotropy, the principal axes are rotated counter-clockwise about the \(x_1, x_2, x_3\)-axis successively by 30\(^\circ\), 45\(^\circ\), and 60\(^\circ\), respectively; the following coefficients are obtained:
Uniform pressure $p = 1$ (N/mm$^2$) is assumed to be applied on top of the cube with side length $S = 2$ m. For the BEM mesh modeling, multiple nodes are applied along edges and at corners. As shown in Fig. 1(b), the BEM modeling employed 136 quadratic quadrilateral elements with total 540 nodes. For numerically integrating the integrals in eq (7a), the Gauss quadrature rule with up to 64 Gaussian points was employed for the highly anisotropic properties. For independent verification, the analysis was also carried out using the finite element software ANSYS, where 57,600 SOLID226 elements were applied (Fig. 1(c)). Fig. 2 shows the displacements and electric potentials calculated at the boundary points along $R_1 = 0.5$ m on the plane $x_3 = 0$. For internal solutions, Fig. 3 plots the von Mises stresses and total electric displacements computed at points along $R_2 = 0.75$ m on the plane $x_3 = 0$. As can be clearly seen from the plots, our BEM results are in quite satisfactory agreements with the FEM analysis.
Figure 2. Boundary solutions of (a) displacements and (b) electric potentials along \( R_1 = 0.5 \) m on \( x_3 = 0 \).

Figure 3. Internal solutions along \( R_2 = 0.75 \) m on \( x_3 = 0 \): (a) von Mises stress, (b) total electric displacement.

Conclusions

For studying the piezoelectric effect in 3D anisotropic solids, this article employs the methodology of dimension expansion to combine all related matrices such that the conventional BIE for dealing with elastostatics may also apply with expanded dimensions. For solving the generalized BIE, the present analysis adopts the fundamental solutions of the 3D anisotropic elasticity formulated by real variable Radon-Stroh formalism [9]. Our successful implementation of the methodology is illustrated by a numerical example, showing satisfactory agreements with the ANSYS analysis. Undoubtedly, our successful implementation has demonstrated its promising extension to more complicated 3D problems.

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References
