Boundary Element for Stretching-Bending Coupling Analysis of Laminated Composite Plates

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Abstract
Among various types of boundary element method (BEM) developed in engineering application, most of the computer programming of BEM is designed for two-dimensional or three-dimensional problems. With the aid of previous studies of Green’s function in the form of Stroh-like complex variable formalism, we can deal with the stretching-bending coupling analysis of a composite laminated plate. An alternative method treating the boundary integral equations is exploited here to eschew the singularities induced within, that is, the outward–moved source points is considered for the present boundary element. In the demonstration of the numerical examples, comparisons are made to various results obtained from analytic solutions, finite element method and the literatures. In the process the present method revealed apparent efficiency without the need to perform the mesh of a domain, and above all it can provide a well-comprehensible scheme and insights of solving the problems of composite laminated plates under diverse boundary conditions using BEM.

Keywords: Boundary element methods, Composite laminates, Stretching-bending coupling

伸縮-彎矩耦合複材疊層板之邊界元素分析

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摘要
在邊界元素法的工程應用中，大多數的程式分析及撰寫僅針對純二維或三維問題著手。隨著複變型態類史搓格林函數的研究與協助，現階段我們已經可以進一步地處理複材疊層板之伸縮-彎矩耦合分析；而為了要避開奇異性問題，在這裡是以一個替代的方式來看待邊界積分式，也就是將源點移至邊界外的方法。在數值結果的呈現上，其比較對象的來源包括了解析解、有限元素法，以及文獻資料。分析過程中，目前的做法不僅顯示出明顯的效率且不需對板內部做網格化的動作，最重要的是它提供了一個易理解的架構和看法，對於利用邊界元素法在多變的邊界條件下求解複材疊層板問題。

關鍵字：邊界元素法、複材疊層板、伸縮-彎矩耦合
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1. Introduction

There are already well-known difficulties in essence which will be encountered when putting the numerical evaluation of boundary integral equations (BIE) in practice for the direct method based on the reciprocal theorem [1]; hence the primary concern goes for the treatment of singularity due to the coincidence of field and source points. The conventional way is to integrate in the sense of Cauchy principal values, which has been adopted and unveiled, in various ways, numerical or mathematical, in the literature on hand; the same methods was also employed in the singular integrations needed for the present BIE and have been represented in Hwu [2,3]. Whereas, some scholars, while in a minority, looked into the alternates to cope with knotty complexities inherent in the BIE, that is to say, to avoid the singularities arose due to some of the terms, whose denominators are of infinite value, in the fundamental solutions; the way is to allocate the source nodes outside of the boundary, while the extent to which it is moved should be examined carefully.

Amid a number of discussions in the literature, a conformable thought[4,5,6,7] were suggested that the outward-moved distance can be chosen as the order of an element length; in addition, Wearing and Bettahar[8] proposed a quantitative description of the range of stable results, a non-dimensional factor \( \lambda \) formulated as the ratio of the distance R between a source and field point to the length L of the corresponding element, that is, \( \lambda = R/L \), as shown in Figure 1. However, regardless of which criterion one would assert, such as the factor \( \lambda \), a suitable distance may be supposed to be multiplex and depends on the problems discussed - say, plates of isotropic materials or composite laminates; it would be likely that divergent results could be obtained owing to inadequate choice of distance. Hence a discussion of numerical stability in the so-called regular BEM, developed with some algorithms, was offered by Ventsel[9].

However, in order to render a more comprehensible and perceptible scheme, the examinations of the extent to which the source movies are essentially carried out by a series of numerical tests utilizing the present BEM in our study; their stable results are accorded with the suggestion concerning the outward-moved distance of the source that described above, that is, \( \lambda = 1 \). Hence in the numerical examples presented here this choice of the distance is taken as the preoccupation in order to provide some further detailed examinations of the results and their comparisons with the other sources and literature. A more comprehensive scheme of the numerical implementation is being prepared for the future journal publication.

2. Boundary integral equations

The boundary integral equations for the coupled stretching-bending analysis of composite laminated plates obtained in [2] is established as

\[
c_{ip}(\xi)u_{p}(\xi) + \int_{\Gamma} c_{ip}(\xi, \mathbf{x})u_{p}(\mathbf{x})d\Gamma(\mathbf{x}) + \sum_{i=0}^{\infty} c_{ip}(\xi, x_{i})u_{p}(x_{i}) = \int_{\Gamma} u_{p}^{*}(\xi, \mathbf{x})q_{i}(\mathbf{x})d\Gamma(\mathbf{x}) + \sum_{i=0}^{\infty} u_{p}^{*}(\xi, x_{i})q_{i}(x_{i}), \quad i, j = 1, 2, 3, 4, p = 1, 2, 3, 4.5,
\]

where \( \xi \) and \( \mathbf{x} \) are the locations of source and field point, respectively; \( \Gamma \) is the boundary of the plate and \( A \) is its surface region. The symbol \( \int_{\Gamma} \) denotes the premise of integrating in the sense of Cauchy principal value; \( N_{c} \) represents the number of corners in general but should exclude the one when the source \( \xi \) is a corner point. Other detailed expressions of the kernels in equation (1) can be found in Hwu[2].

\[
c_{ip}(\xi), \quad i = 1, 2, 3, 4; \quad p = 1, 2, 4, 4.5, \quad \text{are the free term coefficients; instead of adopting the explicit solutions provided in [3] as the source is on the boundary, the free term coefficients will vanish in the present study, i.e. } c_{ip}(\xi) = 0, \text{ due to the allocation of the source being outside the boundary. Hence the further manipulation for the boundary element formulation becomes compact and simple by eluding the possible singularities.}
\]

Now, equation (1) can be written in terms of matrix multiplication in the integrands as:

\[
\begin{align*}
\mathbf{\hat{T}}(\xi, \mathbf{x})u(\mathbf{x})d\Gamma + \sum_{i=0}^{N_{c}} \mathbf{p}_{i}(\xi)u_{i}(x_{i}) &= \mathbf{U}(\xi, \mathbf{x})u(\mathbf{x})d\Gamma + q^{*}(\xi) + \sum_{i=0}^{N_{c}} w_{i}^{*}(\xi)\mathbf{w}_{i}(x_{i}), \\
\end{align*}
\]

where the expressions of \( \mathbf{C}, \mathbf{\hat{T}}, \mathbf{\hat{U}}, \mathbf{p}_{i}, \) and \( \mathbf{w}_{i}^{*} \) are:

\[
\begin{align*}
\mathbf{\hat{T}} &= [u_{ij}(\xi, \mathbf{x})], \quad \mathbf{\hat{U}} = [u_{ij}^{*}(\xi, \mathbf{x})], \\
\mathbf{p}_{i} &= [u_{ij}(\xi, x_{j})], \quad \mathbf{w}_{i}^{*} = [u_{ij}^{*}(\xi, x_{j})], \\
q^{*}(\xi) &= \left\{ \int_{\Gamma} u_{ij}^{*}(\xi, \mathbf{x})q_{i}(\mathbf{x})d\Gamma(\mathbf{x}) \right\}, \\
i = 1, 2, 3, 4, \quad j = 1, 2, 3, 4.
\end{align*}
\]

A conventional linear element with first-order shape
functions is utilized for the discretization; hence the position coordinates \( x \), displacement \( u \), and traction \( t \) at any point on an element can be expressed by its nodal values with the shape functions, \( \sigma_1 \) and \( \sigma_2 \), which are given by

\[
\sigma_i = \frac{1}{2} (1 - \zeta), \quad \sigma_2 = \frac{1}{2} (1 + \zeta),
\]

where \( \zeta \) ranging from -1 to 1 is the non-dimensional coordinate defined by \( \zeta = 2s / \ell_m \), in which \( \ell_m \) is the length of the \( m \)th element and \( s \) is the coordinate lying along this element and directed from the first node to the second node of element \( m \). Then, by similar procedure introduced in [10], equation (2) can be rewritten as

\[
\sum_{n=1}^{N_x} \hat{Y}_n(\xi)u_n + \sum_{l=1}^{N_c} p_l^c(\xi)u_l(x_l) = \sum_{n=1}^{N_y} \hat{G}_n^c(\xi)t_n + q^c(\xi) + \sum_{l=1}^{N_c} w_l^c(\xi)\varepsilon_l(x_l);
\]

in which

\[
\hat{Y}_n(\xi) = \int_{\Gamma_m} T(\xi, x_m^{(1)}, x_m^{(2)}, \zeta)\sigma_1(\zeta)d\Gamma_m(\xi),
\]

\[
\hat{G}_n^c(\xi) = \int_{\Gamma_m} U(\xi, x_m^{(1)}, x_m^{(2)}, \zeta)\sigma_2(\zeta)d\Gamma_m(\xi),
\]

\( i = 1,2 \),

\( \Gamma_m \) denotes the \( m \)th element of the discretized boundary and \( N_x, N_c \) are the number of nodes and corner points. The procedure for solving this formulation is similar to those presented in Hwu[2,3], from which interested readers can acquire the proper treatments.

3. Numerical Examples

The examples presented here are compared with the analytical calculations, literatures, and executive results of FEM commercial software (ANSYS) for validation, where 64 Shell99 quadratic elements are used in the FEM simulations and 8 boundary elements per edge, except for those indicated in the following context, are used for the present BEM. For the convenience of the descriptions in the examples, the regard for the origin of Cartesian coordinate system, being the centers of the plates, is proclaimed beforehand

3.1 A cantilever angle-ply laminated square plate under a uniformly distributed surface load and bending moments

An angle-ply laminated plate [0/45/90/0/-45/-45/90/0/45/90] is subjected to a uniform load of \( 10^3 \) (N/m) and a uniform bending moment of \(-10^2 \) (N-m/m), simultaneously, along one of its edges with the opposite fixed and the other two which are free. The plate dimension is 0.3 m \times 0.3 m \times 5 mm; each layer of the laminates is made of fiber-reinforced composite, of which the mechanical properties are \( E_1 = 181 \) GPa, \( E_2 = 10.3 \) GPa, \( G_{12} = 7.17 \) GPa, v = 0.28. By using the present BEM, the deflection at the plate center is 1.4663 mm, which is close to that by ANSYS, 1.4747 mm; the deflection contours by both methods are also shown in Figure 2, in which the black curves indicate our BEM results.

3.2 A simply-supported laminated plate under a uniformly distributed surface load with its in-plane dimensions varied according to width-to-height ratio (aspect ratio)

This case is to demonstrate the relation between normalized deflection \( \bar{w} \) [11,12] and plate aspect ratio, \( AR = a/b, a, b \) is the in-plane dimension of the plate along \( x, y \) direction, respectively, for a simply-supported laminated plate applied by a uniform surface load. The material properties used are \( E_1/E_2 = 25, G_{12} = 0.5 E_2 \), and \( v_{12} = 0.25 \). The normalized deflection \( \bar{w} \) is expressed as \( w_0 \cdot E_2 \cdot h^2 / (q \cdot b^4) \), where \( w_0 \) is the deflection at the plate center; \( h \) is the plate thickness; \( q \) is the value of surface load. The results are compared with those of FEM and Reddy[12], according to three kinds of fiber orientation, namely, [0/90]s, [0/90]2, and [0/90], respectively, with AR varying from 0.25 to 5.0, and shown in Figures 3-5. We can observe that the solutions of unsymmetric laminate provided by Reddy, obtained by series approximations, are much away from the BEM and FEM results in Figure 4 and Figure 5, specifically those of [0/90] orientation. Furthermore, the bending-stretching coupling effects, demonstrate the increasing deflections in anti-symmetric cross-ply laminates[13] represented here; the effects decrease as the number of layers increase, as shown in Figure 6, together with the curve of [0/90]. Note that all laminates have identical total thickness.

3.3 Validation with analytical method utilizing composite laminates

It is well known that the lay-up of unsymmetric laminates will result in the deformation of a plate, under, either or both, the circumstances of pure tension or pure bending, which reforms a dis-symmetric appearance due to coupling effects. An analysis of this problem can be achieved by considering the analytical method utilizing the
4. Conclusions

For the problems of stretching-bending coupling composite laminates, this study uses the analytic solution for infinite domain as the Green's function, along with the boundary element method having the source located outside the domain, to evaluate the deformation of plates made of different kinds of material under various types of boundary conditions. The results are not only consistent with analytic solutions available, those of finite elements simulation and other BEM applications, but also display the accuracy and rapidity of convergence.

Overall, the sewing up of present coupling problems for laminated plates in BEM, with outward-moved source, can establish the analyses of cases with concurrent applied stretching and out-of-plane bending, as well as much complicated boundary conditions. The purpose of this study is to provide a way based on boundary element methods for future developments, say, works relevant to coupling effects, specifically and significantly, the applications for involvements with holes, cracks, inclusions and piezoelectric materials in composite laminates.

Reference

Table 1: Comparison of strain, curvatures and resultant forces at the center using analytic solution, present BEM, and error %.

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<th>present BEM</th>
<th>error %</th>
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Figure 1: Positions of the outward-moved source points.

Figure 2: Deflection contours of the cantilever angle-ply laminated plate.

Figure 3: Normalized deflection at the plate center versus plate aspect ratio.

Figure 4: Normalized deflection at the plate center versus plate aspect ratio.

Figure 5: Normalized deflection at the plate center versus plate aspect ratio.

Figure 6: Normalized deflection at the plate center versus plate aspect ratio.