MATRiX POWER FUNCTION SOLUTIONS FOR
MULti-MATERIAL INTERFACE CORNERS

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ABSTRACT: In mathematical modeling of engineering problems, most of the physical
behaviors are first studied through simple models with scalar quantities. To preserve the
characteristics of simple models and to reflect the complexity of real problems, it becomes
necessary to take the matrix form expressions in practical mathematical modeling. For
example, a scalar form \( f = kx \) is generally used to describe the physical behavior of elastic
springs, whereas for general elastic bodies we need to consider the matrix form expression
\( f = Kx \). The field of fracture mechanics starts from the near tip solutions of cracks in
homogeneous isotropic elastic materials, which are usually written in scalar form. Through the
scalar form near tip solutions of cracks, it is easily to understand the singular behavior of stresses and to define the proper stress intensity factors. When the cracks become interface
corners between two dissimilar anisotropic materials, due to the complexity raised by the
variations of corner angles and material properties it becomes desirable to consider the near tip
solutions be expressed in matrix form. To fulfill the need for generalization to general fracture
problems, recently we proposed a matrix power function form solution for the near tip field of
multi-material interface corners. Here, we like to re-examine this matrix form solution and use
this solution to define the stress intensity factors for general interface corners. This new
definition of stress intensity factor is applicable for all possible singular orders – repeated or
distinct, real or complex, and keeps the same unit for all possible interface corners. Therefore,
it will be helpful for bridging the problems of cracks, corners, interface cracks and interface
corners.

KEYWORDS: interface corner, interface crack, multi-material wedge, singular order, stress
intensity factor, near tip solution, matrix power function.

INTRODUCTION

Based on the semi-analytical solution obtained from the scaled boundary finite element method,
Song, et al. (2010) proposed an interesting matrix power function for the near tip solutions of
multi-material wedges. The solutions expressed by the matrix power function can be written as
\[
\sigma(r, \theta) = \frac{1}{2\pi \ell} (r / \ell)^{-\Delta(\theta)} k(\theta)
\]
where \( \sigma(r, \theta) \) is a vector of stresses at point \((r, \theta)\), \( \Delta(\theta) \) is called the matrix of singular orders at angle \( \theta \), \( k(\theta) \) is a vector of stress intensity factors along the radial direction \( \theta = \text{constant} \), and \( \ell \) is a reference length. In Song’s study (2010), the matrix of singular order \( \Delta(\theta) \) is studied based upon the scaled boundary finite element method and therefore no explicit analytical solution is written for this important matrix. Because this matrix form
solution preserves the characteristics of scalar form solution for cracks and can cover all the
possible situations of interface corners, it deserves to have a further analytical study on this
matrix form near tip solution. Therefore, in our recent paper (Hwu, 2011) most of the efforts were put on the analytical derivation of the matrix power function form near tip solutions. Since this matrix power function form is valid for all possible cases of interface corners, all kinds of singularities including the negative power exponential and the logarithmic singularity were discussed.

**EXPLICIT SOLUTIONS FOR THE MATRIX POWER FUNCTION**

In the matrix power function near tip solution (1), the stress vector \( \sigma(r, \theta) \) is a vector composed of the traction along \( \theta = \text{constant} \), i.e.,

\[
\sigma(r, \theta) = \begin{bmatrix}
\sigma_{\theta\theta}(r, \theta) \\
\sigma_{\theta\phi}(r, \theta) \\
\sigma_{\phi\phi}(r, \theta)
\end{bmatrix}
\]  

(2)

The explicit solutions of the matrix of singular orders \( \Delta(\theta) \) have been obtained as follows. (Hwu, 2011)

(i) If no matter the singular orders \( \delta_\alpha, \alpha = 1,2,3 \), are real or complex, repeated or distinct, their associated eigenfunctions \( \lambda_\alpha(\theta), \alpha = 1,2,3 \) are independent each other, then

\[
\Delta(\theta) = \Lambda^\ast(\theta) < \delta_\alpha > \Lambda^{-1}(\theta), \quad \Lambda^\ast(\theta) = \Omega(\theta)\Lambda(\theta)
\]  

(3)

where \( \Omega(\theta) \) and \( \Lambda(\theta) \) are, respectively, the rotation matrix and the matrix of stress eigenfunction defined by

\[
\Omega(\theta) = \begin{bmatrix}
cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad \Lambda(\theta) = [\lambda_1(\theta) \lambda_2(\theta) \lambda_3(\theta)].
\]  

(4)

The angular bracket \( <> \) used in (3) stands for a diagonal matrix in which each component is varied according to the subscript \( \alpha \), e.g., \( < z_\alpha > \text{ diag. } [z_1, z_2, z_3] \).

(ii) If one of the singular orders \( \delta_\alpha, \alpha = 1,2,3 \), is a double root and no enough independent eigenfunctions exist, i.e., if \( \delta_1 = \delta_2, \lambda_1(\theta) = \lambda_2(\theta) \), then

\[
\Delta(\theta) = \hat{\Lambda}^\ast(\theta) < \delta_\alpha > \hat{\Lambda}^{-1}(\theta), \quad \hat{\Lambda}^\ast(\theta) = \Omega(\theta)\hat{\Lambda}(\theta),
\]  

(5)

where

\[
\hat{\Lambda}(\theta) = [\lambda_1(\theta) \dot{\lambda}_1(\theta) \lambda_3(\theta)].
\]  

(6)

(iii) If one of the singular orders \( \delta_\alpha, \alpha = 1,2,3 \), is a triple root and no enough independent eigenfunctions exist, i.e., if \( \delta_1 = \delta_2 = \delta_3, \lambda_1(\theta) = \lambda_2(\theta) = \lambda_3(\theta) \), then

\[
\Delta(\theta) = \ddot{\Lambda}^\ast(\theta) < \delta_\alpha > \ddot{\Lambda}^{-1}(\theta), \quad \ddot{\Lambda}^\ast(\theta) = \Omega(\theta)\ddot{\Lambda}(\theta),
\]  

(7)

where

\[
\ddot{\Lambda}(\theta) = [\lambda_1(\theta) \dot{\lambda}_1(\theta) \ddot{\lambda}_1(\theta)].
\]  

(8)

In the above, the over dot means differentiation with respect to the singular order, i.e.,

\[
\dot{\lambda}_1(\theta) = \frac{\partial}{\partial \delta_1} \{\lambda_1(\theta)\}, \quad \ddot{\lambda}_1(\theta) = \frac{\partial^2}{\partial \delta_1^2} \{\lambda_1(\theta)\}.
\]  

(9)
The symbols with the forms of $< f_{\alpha} >$ and $< f_{\mu} >$ are defined as

$$
< f_{\alpha} > := \begin{bmatrix}
    f_1 \frac{\partial f_1}{\partial \delta_1} & 0 \\
    0 & f_1 \\
    0 & 0 & f_3
\end{bmatrix},
< f_{\mu} > := \begin{bmatrix}
    f_1 \frac{\partial f_1}{\partial \delta_1} \frac{\partial^2 f_1}{\partial \delta_1^2} & 0 \\
    0 & f_1 \frac{2 \partial f_1}{\partial \delta_1} \\
    0 & 0 & f_1
\end{bmatrix}.
$$ (10)

Therefore,

$$
< \delta_{\alpha} > := \begin{bmatrix}
    \delta_1 & 1 & 0 \\
    0 & \delta_1 & 0 \\
    0 & 0 & \delta_3
\end{bmatrix},
< \delta_{\mu} > := \begin{bmatrix}
    \delta_1 & 1 & 0 \\
    0 & \delta_1 & 2 \\
    0 & 0 & \delta_1
\end{bmatrix}.
$$ (11)

In matrix operation, it is known that if $f(\Lambda) = \sum_{m=0}^{\infty} c_m \Lambda^m$ converges, and if $\Lambda$ is similar to a diagonal matrix, such as $\Lambda^{-1} \Lambda^* = < \delta_{\alpha} >$ shown in (3), then $f(\Lambda) = \Lambda^* < f(\delta_{\alpha}) > \Lambda^{-1}$. With this understanding, the matrix power function $(r / \ell)^{-\Lambda(\theta)}$ given in (1) can be calculated by

$$
(r / \ell)^{-\Lambda(\theta)} = \begin{cases}
    \Lambda^*(\theta) < (r / \ell)^{-\delta_{\alpha}} > \Lambda^{-1}(\theta), & \text{for case (i)}, \\
    \Lambda'(\theta) < (r / \ell)^{-\delta_{\alpha}} > \Lambda^{-1}(\theta), & \text{for case (ii)}, \\
    \Lambda'(\theta) < (r / \ell)^{-\delta_{\alpha}} > \Lambda^{-1}(\theta), & \text{for case (iii)}.
\end{cases}
$$ (12)

DEFINITION OF THE STRESS INTENSITY FACTOR

With the matrix power function form solution (1), the stress intensity factors $k(\theta)$ can be defined as

$$
k(\theta) = \lim_{r \to 0} \sqrt{2\pi \ell} (r / \ell)^{\Lambda(\theta)} \sigma(r, \theta),
$$ (13)

where

$$
k(\theta) = \begin{bmatrix}
    K_{II}(\theta) \\
    K_{I}(\theta) \\
    K_{III}(\theta)
\end{bmatrix},
$$ (14)

and $K_{I}(\theta), K_{II}(\theta), K_{III}(\theta)$ are, respectively, the stress intensity factors of opening mode, shearing mode, and tearing mode. When $\theta = 0$ which is the conventional definition for a crack in homogeneous materials, we have

$$
k = \lim_{r \to 0} \sqrt{2\pi \ell} (r / \ell)^{\Lambda} \sigma(r, 0),
$$ (15)

where $k = k(0), \Delta = \Delta(0)$.

From the explicit solutions shown in the previous Section, we see that the definitions written as (13) and (15) are valid for all possible cases of interface corners including the one with logarithmic singularity. Since the stress intensity factors $k(\theta)$ defined in (13) will vary according to the selected direction $\theta = \text{constant}$, like the principal stress which occurs at the plane where the shear stress vanishes, the maximum stress intensity factor of opening mode - the principal stress intensity factor represented by $K_I^p$, may occur at the radial direction $\theta = \theta_p$. 
where
\[ K_{II}(\theta_p) = 0, \quad \text{and} \quad K^p = K_f(\theta_p) = \max\{K_f(\theta)\}. \quad (16) \]

The concept of the principal stress intensity factor was proposed in (Hwu, 2011) and has been proved to be equivalent to the concept of maximum tangential stress criterion (Erdogan and Sih, 1963) when it was employed to the fracture prediction.

**SPECIAL CASES**

**Cracks in homogeneous materials**

When a crack is located in a homogeneous material, either isotropic or anisotropic, it has been shown that (Hwu, 2011)
\[ \delta_1 = \delta_2 = \delta_3 = 0.5, \quad \Lambda = I, \quad (17) \]
where \( I \) is an identity matrix. With (17), from (12) we have \((r/\ell)^{-\Lambda(r)} = (r/\ell)^{-0.5}\), and hence the definition (15) reduces to the conventional definition
\[ k = \lim_{r \to 0} \sqrt{2\pi r} \sigma(r, 0). \quad (18) \]

**Interface cracks between two dissimilar orthotropic materials**

When a crack is located on the interface between two dissimilar orthotropic materials, the singular orders \( \delta_{\alpha}, \alpha = 1,2,3 \) and their associated eigenfunction matrix \( \Lambda = [\lambda_1, \lambda_2, \lambda_3] \) have been obtained by Ting (1986) and Hwu (1993) as
\[ \delta_1 = 0.5 + i \epsilon, \quad \delta_2 = 0.5 - i \epsilon, \quad \delta_3 = 0.5, \]
\[ \Lambda = \begin{bmatrix} -i/\sqrt{2D_{11}} & i/\sqrt{2D_{11}} & 0 \\ 1/\sqrt{2D_{22}} & 1/\sqrt{2D_{22}} & 0 \\ 0 & 0 & 1/\sqrt{D_{33}} \end{bmatrix}, \quad (19) \]

where \( \epsilon \) is the oscillatory index which characterizes the oscillatory behavior of the stresses near the crack tip, and \( D_{ij}, \ i, j = 1,2,3, \) are the components of the bi-material matrix \( D \). The explicit expressions of \( \epsilon \) and \( D_{ij} \) for the general anisotropic bi-materials can be found in (Hwu, 2010). For isotropic bi-materials under plane strain condition, it has been shown that (Hwu, 1993)
\[ D_{11} = D_{22} = \frac{2(1-\nu_1^2)}{E_1} + \frac{2(1-\nu_2^2)}{E_2}, \quad D_{33} = \frac{2(1+\nu_k)}{E_1} + \frac{2(1+\nu_2)}{E_2}, \quad (20) \]

where \( E_k, \nu_k, \ k = 1,2, \) are Young’s modulus and Poisson’s ratio of material 1 and 2.

With the results of (19), the matrix of singular orders \( \Lambda \) defined in (3) can be obtained as
\[ \Lambda = \begin{bmatrix} 1/2 & \epsilon \sqrt{D_{22}/D_{11}} & 0 \\ -\epsilon \sqrt{D_{11}/D_{22}} & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}. \quad (21) \]

The matrix power function \((r/\ell)^\Lambda\) becomes
\[ (r/\ell)^{\lambda} = (r/\ell)^{1/2} \begin{bmatrix} c^*(r) & s^*(r)\sqrt{D_{22}/D_{11}} & 0 \\ -s^*(r)\sqrt{D_{11}/D_{22}} & c^*(r) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \] (22)

where
\[ c^*(r) = \cos[\varepsilon \ln(r/\ell)], \quad s^*(r) = \sin[\varepsilon \ln(r/\ell)]. \] (23)

The definition (15) then reduces to
\[
K_I = \lim_{r \to 0} \sqrt{2\pi r} \left\{ -\sqrt{D_{11}/D_{22}} \sin[\varepsilon \ln(r/\ell)]\sigma_{\theta\theta}(r,0) + \cos[\varepsilon \ln(r/\ell)]\sigma_{\theta\phi}(r,0) \right\},
\]
\[
K_{II} = \lim_{r \to 0} \sqrt{2\pi r} \left\{ \cos[\varepsilon \ln(r/\ell)]\sigma_{\theta\phi}(r,0) + \sqrt{D_{22}/D_{11}} \sin[\varepsilon \ln(r/\ell)]\sigma_{\theta\theta}(r,0) \right\},
\] (24)
\[
K_{III} = \lim_{r \to 0} \sqrt{2\pi r} \sigma_{\phi\phi}(r,0),
\]
that can be further reduced to the one proposed by Rice (1988) for isotropic bi-material interface cracks whose \( D_{11}/D_{22} = 1 \).

**Interface cracks between two dissimilar anisotropic materials**

With \( \mathbf{\Xi}(0) = \mathbf{I} \), we have \( \mathbf{\Lambda}'(0) = \mathbf{A}(0) = \mathbf{A} \). Also, \( \delta_1 = 0.5 + i\varepsilon, \quad \delta_2 = 0.5 - i\varepsilon \), and \( \delta_3 = 0.5 \) for general anisotropic bi-materials. With these results, use of (12)\textsubscript{1} will lead the definition (15) to
\[
k = \lim_{r \to 0} \sqrt{2\pi r} \mathbf{A} < (r/\ell)^{-i\varepsilon} > \mathbf{A}^{-1} \mathbf{\sigma}(r,0),
\] (25)

which is equivalent to the one proposed by Hwu (1993).

**INVESTIGATION THROUGH NUMERICAL EXAMPLES**

Consider an interface corner as shown in Figure 1. The reference length \( \ell \) is selected to be 10 mm. The materials above and below the interface are, respectively, isotropic and orthotropic, whose properties are

- **Isotropic:** \( E = 10\text{GPa}, \quad \nu = 0.2 \)
- **Orthotropic:** \( E_{11} = 134.45\text{GPa}, \quad E_{22} = E_{33} = 11.03\text{GPa} \)
  \( G_{12} = G_{13} = 5.84\text{GPa}, \quad G_{23} = 2.98\text{GPa} \)
  \( \nu_{12} = \nu_{13} = 0.301, \quad \nu_{23} = 0.49 \).

![Figure 1. An interface corner between two dissimilar material](image)

\( d/W = 1/3, \quad h/W = 1/15, \quad d/L = 1/18, \quad \sigma = 10\text{MPa} \).
As presented in our previous study (Hwu and Huang, 2011), Figure 2 shows the first three singular orders for interface corners ranging from $\Delta \theta_1 = 20^\circ$ to $180^\circ$. It is interesting to see that the singular orders meet the type change at $\Delta \theta_1 = 64.670^\circ$ and $\Delta \theta_1 = 140.597^\circ$. Figure 3 shows the results of stress intensity factors versus angle of interface corner $\Delta \theta_1$. From this Figure, we see that no abrupt change occurs in the entire region for the stress intensity factors defined by (15), represented here by the notation $K^D_I$ and $K^D_{II}$, whereas the stress intensity factors defined in (Hwu and Kuo, 2007) represented by $K^C_I$ and $K^C_{II}$ meet an abrupt change at the transition angle $\Delta \theta_1 = 140.597^\circ$. Detailed explanation about the difference of their values can be found in (Hwu and Huang, 2011).

![Figure 2. Variation of singular orders $\delta_{\alpha}$ on corner angle $\Delta \theta_1$.](Hwu and Huang, 2011)

![Figure 3. Variation of stress intensity factors $K_I$ and $K_{II}$ on corner angle $\Delta \theta_1$.](Hwu and Huang, 2011)
CONCLUSIONS

From the analytical expressions and numerical examples shown in this paper we see that the matrix power function not only provides a compact matrix form near tip solution, but also gives a unified definition for the stress intensity factors valid for all possible multi-material interface corners. Furthermore, this newly defined stress intensity factor has a unified unit (Pa√m) and will vary smoothly even when the stress singularity changes among real, complex and logarithmic types.

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