Boundary Integral Equations for Unsymmetric Laminated Composites

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Abstract. With the aid of the reciprocal theorem of Betti and Raleigh, recently the boundary integral equations for the stretching-bending coupling analysis of general laminated plates was derived. To make the boundary integral equations work for the programming of boundary element codes, the appropriate fundamental solutions derived from the Green’s function for an infinite laminated plate subjected to concentrated forces/moments has also been obtained by using the Stroh-like complex variable formalism. The free term coefficients of the boundary integral equations were obtained explicitly and expressed in real form. Alternative formulae for calculating the free term coefficients were also derived using five rigid body movements. In this paper, the fundamental solutions for specific problems such as unsymmetric laminates containing holes, cracks or inclusions will be derived. With these fundamental solutions, the boundary integral equation for unsymmetric laminated composites can be further improved to analyze the coupled stretching-bending laminated plates with holes/cracks/inclusions.

Introduction

The main feature of boundary integral equation is that only the boundaries of the region being investigated are involved, which therefore leads to much smaller systems of algebraic equations for final solution than other methods such as finite element method. Based upon the boundary integral equations for different problems, several different boundary elements used in solid mechanics or fluid mechanics have been developed [1]. Among these boundary elements, the development for plate bending problems started relatively later than that for two-dimensional or three-dimensional elastostatics. As to the stretching-bending coupling analysis of composite laminates, the associated boundary integral equation was just derived recently by Hwu [2] with the aid of the reciprocal theorem of Betti and Raleigh. The basic variables considered in the formulation are inplane displacements or tractions in x and y directions, deflection or effective transverse shear force, normal slope or bending moment, and corner force.

To make the boundary integral equations work for the programming of boundary element codes, one of the important tasks is finding the appropriated fundamental solutions, which are associated with Green’s functions, i.e., solutions for problems subjected to concentrated forces and moments at arbitrary points. Recently, by employing Stroh-like formalism for the coupled stretching-bending analysis of composite laminates [3,4], several Green’s functions have been obtained such as those for infinite laminates [5], laminates with holes/cracks [6], and laminates with inclusions [7]. To establish the boundary integral equations for general laminates with coupled stretching-bending deformation, in this paper the fundamental solutions are derived based upon the Green’s functions for infinite laminates obtained in [5-7].

Boundary Integral Equations

The reciprocal theorem of Betti and Raleigh in terms of stresses and strains can be expressed as

$$\int_{\Omega} \sigma_{ij} \varepsilon_{ij} \, d\Omega = \int_{\Omega} \sigma_{ij}^* \varepsilon_{ij} \, d\Omega, \quad (1)$$

where $\sigma_{ij}, \varepsilon_{ij}$ and $\sigma_{ij}^*, \varepsilon_{ij}^*$, $i,j = 1,2,3$, are the stresses and strains induced by two different loading systems on the same elastic body whose region is denoted by $\Omega$. In the boundary integral equations derived from (1), the symbols without superscript * denote the physical quantities of the actual system, whereas the symbols with superscript * denote the physical quantities of the complementary system loaded by a concentrated force or moment. If the concentrated force or moment is applied on the boundary of the system, some physical quantities such as the stresses will become singular when the nodal point approaches the loading point. Since the boundary element is constructed through the boundary integral equations on the
boundary nodes, to avoid the singular problem caused by the concentrated forces or moments it is important to have the integral equations valid for all boundary points. With this consideration, a general boundary integral equation of laminates with coupled stretching-bending deformation (see Fig. 1) valid for any location such as internal, external, boundary or corner points has been obtained as [2]

\[
c_j(\xi)u_j(\xi) + c_{i5}(\xi)\beta_i(\xi) + \int_{\Gamma} f'_i(\xi,x)u_j(\chi)d\Gamma(\chi) + \sum_{k=1}^{N_c} t'_{ik}(\xi,x_k)u_j(x_k)
\]

\[
= \int_{\Gamma} u'_i(\xi,x)t_j(\chi)d\Gamma(\chi) + \int_{\Gamma} u'_i(\xi,x)q_j(\chi)d\Gamma(\chi) + \sum_{k=1}^{N_c} u'_i(\xi,x_k)t_j(x_k), \quad i, j = 1, 2, 3, 4,
\]

where \(c_y, i = 1, 2, 3, 4, j = 1, 2, 3, 4, 5\) are the free term coefficients whose determination will be described later; \(\int_{\Gamma} \) denotes the integral taken in the sense of Cauchy principal value; \(N_c\) is the number of corners excluding the one which coincides with the source point; \(u_1, u_2\) and \(u_3\) are the midsurface displacements in \(x, y\) and \(z\) directions; \(u_4 = \beta_n\) is the slope in normal direction; \(\beta_i\) is the slope in tangential direction; \(t_1, t_2\) are the tractions in \(x\) and \(y\) directions; \(t_3 = V_n\) and \(t_4 = M_n\) are the effective transverse shear force and bending moment on the surface with normal \(n\); \(q_1, q_2\) and \(q_3\) are the distributed loads in \(x, y\) and \(z\) directions; \(q_4 = m_n\) is the distributed moment applied on the surface with normal \(n\); \(t_c = M_{n+} - M_{n-}\) is the corner force. \(u'_i(\xi,x)\), \(t'_i(\xi,x)\) and \(t'_{ik}(\xi,x_k)\), \(i = 1, 2, 3, j = 1, 2, 3, 4\), represent, respectively, \(u_j\), \(t_j\) and \(t_k\) at point \(x\) corresponding to a unit point force acting in the \(e_j\) direction applied at point \(\xi\), whereas \(u'_{ij}(\xi,x)\), \(t'_{ij}(\xi,x)\) and \(t'_{ijk}(\xi,x_k)\), \(j = 1, 2, 3, 4\) represent \(u_j\), \(t_j\) at point \(x\) corresponding to a unit point moment acting on the surface with normal \(n\) applied at point \(\xi\). The normal \(n\) associated with the boundary \(\Gamma(x)\) is the normal of the surface boundary at point \(x\), while the normal \(n\) associated with the interior domain \(A(x)\) is the normal compatible with the direction of the distributed moment \(q_n(x)(= m_n(x))\) at point \(x\). For example, if a plate is subjected to a uniform moment expressed by \(m_n\) and \(m_j\), an equivalent expression with \(m_4 = 0, m_j \neq 0\) should be found to determine the direction \(n\) of the applied moment.

In (2), the free term coefficients \(c_y, i = 1, 2, 3, 4, j = 1, 2, 3, 4, 5\) can be computed by considering the following five different rigid body movements:

i. \(u_1 = 1, u_2 = u_3 = \beta_n = \beta_i = 0\), for all \(x\),
ii. \(u_2 = 1, u_1 = u_3 = \beta_n = \beta_i = 0\), for all \(x\),
iii. \(u_3 = 1, u_1 = u_2 = \beta_n = \beta_i = 0\), for all \(x\),
iv. \(\beta_n = 1, u_1 = u_2 = \beta_i = 0\), for all \(x\), \(u_3(\xi) = 0\),
v. \(\beta_i = 1, u_1 = u_2 = \beta_n = 0\), for all \(x\), \(u_3(\xi) = 0\).

Knowing that \(t_j = q_j = t_c = 0\), \(j = 1, 2, 3, 4\) for rigid body movements, substitution of (3) into the boundary integral equations (2) will then give us

\[
c_{i1}(\xi) = -\int_{\Gamma} t'_{i1}(\xi,x)d\Gamma(\chi),
\]
\[
c_{i2}(\xi) = -\int_{\Gamma} t'_{i2}(\xi,x)d\Gamma(\chi),
\]
\[
c_{i3}(\xi) = -\int_{\Gamma} t'_{i3}(\xi,x)d\Gamma(\chi) - \sum_{k=1}^{N_c} t'_{ik}(\xi,x_k),
\]
\[
c_{i4}(\xi) = -\int_{\Gamma} [f'_i(\xi,x)u_{14}(\chi) + t'_i(\xi,x)\cos(\alpha - \theta)]d\Gamma(\chi) - \sum_{k=1}^{N_c} t'_{ik}(\xi,x_k)u_{14}(x_k),
\]
\[
c_{i5}(\xi) = -\int_{\Gamma} [f'_i(\xi,x)u_{15}(\chi) + t'_i(\xi,x)\sin(\alpha - \theta)]d\Gamma(\chi) - \sum_{k=1}^{N_c} t'_{ik}(\xi,x_k)u_{15}(x_k), \quad i = 1, 2, 3, 4,
\]

where
\[ u_{x_1}(x) = (x_1 - \xi_1) \sin \alpha - (x_2 - \xi_2) \cos \alpha, \]
\[ u_{x_2}(x) = -(x_1 - \xi_1) \cos \alpha - (x_2 - \xi_2) \sin \alpha, \]

and \( \theta \) is the angle directed counterclockwise from \( x \)-axis to the tangent direction \( s \) of point \( x = (x_1, x_2) \), and \( \alpha \) is the value of direction angle \( \theta \) at the source point \( \xi = (\xi_1, \xi_2) \).

In addition to the computation through rigid body movement, the explicit expressions of the free term coefficients have also been obtained in [2], which are the integrals around the source point \( \xi \) instead of the integrals around the whole boundary of the body given in (4). The integration for special laminates such as the isotropic plates has also been performed and the results have been proved to be identical with the well known uncoupled in-plane and pure bending cases, i.e., \( c_{ij} = \delta_{ij} / 2 \) for a smooth boundary, \( c_{ij} = \delta_{ij} \) for an internal point, in which \( \delta_{ij} \) is the Kronecker delta, and \( c_{ij} = 0 \) for a point outside the body.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Laminate geometry, stress resultants and bending moments. [5]}
\end{figure}

**Fundamental Solutions**

(i) **Infinite laminates**

Consider an infinite laminate subjected to a concentrated force and moment at point \( \xi = (\xi_1, \xi_2) \). The elasticity solution of this problem is generally called *Green’s function* and has been obtained in [5] by using Stroh-like formalism. Based upon the Green’s function obtained in [5], the fundamental solutions \( \tau_{ij}(\xi, x) \), \( u_{ij}(\xi, x) \) and \( t_{ij}(\xi, x) \) were obtained as [2]
\[ u^*_i(\xi, x) = 2 \text{Re} \{i^T Af(\xi, x)\}, \]
\[ u^*_{i2}(\xi, x) = 2 \text{Re} \{i^T Af(\xi, x)\}, \]
\[ u^*_i(\xi, x) = -2 \text{Re} \{i^T Af(\xi, x)\}, \]
\[ u^*_{i4}(\xi, x) = 2 \text{Re} \{n^*_{i}(x)Af(\xi, x)\}, \]
\[ f^*_i(\xi, x) = -2 \text{Re} \{n^*_{i}(x)B<\mu_a > f'(\xi, x)\}, \]
\[ f^*_{i2}(\xi, x) = 2 \text{Re} \{n^*_{i}(x)Bf'(\xi, x)\}, \]
\[ f^*_{i3}(\xi, x) = 2 \text{Re} \{s^*_{i}(x)B< s'_i + s'_2 \mu_a > f'(\xi, x)\} \]
\[ + 2 \text{Re} \{s^*_{i2}(x)B< n_2 - n_1 \mu_a > f'(\xi, x)\} \]
\[ + 2 \text{Re} \{s^*_{i2}(x)B< n_2 - n_1 \mu_a > f'(\xi, x)\}, \]
\[ f^*_{i4}(\xi, x) = 2 \text{Re} \{n^*_{i}(x)B< n_2 - n_1 \mu_a > f'(\xi, x)\}, \]
\[ f^*_{i5}(\xi, x) = 2 \text{Re} \{n^*_{i}(x)Bf'(\xi, x)\} \]
\[ i = 1, 2, 3, 4. \]

In the above, \( R \) stands for the real part of a complex number; the angular bracket \(<\cdot\>\) stands for a diagonal matrix in which each component is varied according to the subscript \( \alpha \); \( A \) and \( B \) are two \( 4 \times 4 \) complex matrices which are called material eigenvector matrices; \( \mu_{a, \alpha} = 1, 2, 3, 4 \) are the material eigenvalues with positive imaginary part; \( i, j, k = 1, 2, 3 \) are the \( 4 \times 1 \) unit base vectors whose \( k \)th component is unity and all the others are zero; \( s_i \) and \( s_3 \) are the first and second component of tangent vector \( s \), and \( n_i \) and \( n_2 \) are the first and second component of normal vector \( n \). They are related to the direction angle \( \theta \) by
\[ n^T = (-\sin \theta, \cos \theta), \quad s^T = (\cos \theta, \sin \theta). \]

As to \( s'_i \) and \( s'_2 \), they can be calculated by
\[ s'_i = \frac{\partial s_i}{\partial \theta} \quad \text{and} \quad s'_2 = \frac{\partial s_2}{\partial \theta}, \]
\[ (5c) \]
where \( R \) is the radius of curvature of the considered point. If the point is located on a straight boundary, \( R \to \infty \) and \( s'_i = s'_2 = 0 \). In (5a), the vectors \( n_p, s_p, s'_i, n_i, s_i \) related to the normal \( n \) and tangent \( s \) are defined by
\[ n_p = \begin{pmatrix} n_1 \\ n_2 \\ 0 \\ 0 \end{pmatrix}, \quad s_p = \begin{pmatrix} 0 \\ 0 \\ n_1 \\ n_2 \end{pmatrix}, \quad s'_i = \begin{pmatrix} s'_1 \\ s'_2 \end{pmatrix}, \quad n_i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 - n_1^2 - n_2^2 \end{pmatrix}, \quad s_i = \begin{pmatrix} -n_2 \\ -n_1 \\ n_1 - 1/2 \\ -n_2 \end{pmatrix}, \]
\[ (5d) \]
\[ n_c = n^*_p - n^*_i, \quad s_c = s^*_p - s^*_i. \]

The complex functions \( f_k(\xi, x), k = 1, 2, 3, 4 \) related to the Green’s functions are
\[ f_k(\xi, x) = \frac{1}{2\pi i} \ln(z_\alpha - \hat{z}_\alpha) > A^T i, \]
\[ f_2(\xi, x) = \frac{1}{2\pi i} \ln(z_\alpha - \hat{z}_\alpha) > A^T i, \]
\[ f_3(\xi, x) = \frac{1}{2\pi i} \ln(\zeta - \hat{z}_\alpha) > A^T i, \]
\[ f_4(\xi, x) = \frac{1}{2\pi i} \ln(z_\alpha - \hat{z}_\alpha)[\ln(z_\alpha - \hat{z}_\alpha) - 1] > A^T i, \]
\[ (6a) \]
where
\[ z_\alpha = x_1 + \mu_a x_2, \quad \hat{z}_\alpha = \xi_1 + \mu_a \xi_2, \quad \alpha = 1, 2, 3, 4. \]
\[ (6b) \]
In the above, the subscripts \( p, b, t \) and \( c \) denote, respectively, the values related to the inplane, bending, twisting and corner responses. The superscript \( T \), and \( + \) and \( - \) stands for the transpose of a matrix or vector, and the values ahead of and behind the corner. The normal and tangential vectors, \( n \) and \( s \), will depend on
the location of \( x \) or \( \xi \). The prime (\( \cdot \)), double prime (\( \cdot \cdot \)) and the overtilde (\( \tilde{\cdot} \)) denote, respectively, the first derivative, second derivative and the integral, for example,

\[
\begin{align*}
 f'(\xi, x) &= \frac{1}{2\pi i} (z_a - \tilde{z}_a)^{-1} > A^T i_1, \\
f''(\xi, x) &= \frac{-1}{2\pi i} (z_a - \tilde{z}_a)^{-2} > A^T i_1, \\
 \tilde{f}(\xi, x) &= \frac{1}{2\pi i} (z_a - \tilde{z}_a)[\ln(z_a - \tilde{z}_a) - 1] > A^T i_1.
\end{align*}
\] (6c)

(ii) Laminates with holes or cracks

Similar to the case discussed previously for the infinite laminates, the fundamental solutions for laminates with holes or cracks can be found through the use of the Green’s functions obtained in [6]. By following the steps described in [2], we obtain the fundamental solutions for laminates with holes or cracks, which can also be expressed by (5a-d) in which (\( i = 1, 2, 3, 4 \), are as follows.

\[
\begin{align*}
 f_1(\xi, x) &= F_{1i}, \\
f_2(\xi, x) &= F_{2i}, \\
f_3(\xi, x) &= (F_{31} + F_{32} + F_{33} + F_{34} + F_{35} + F_{36}) i_3, \\
f_4(\xi, x) &= F_{n_4}(\xi),
\end{align*}
\] (7a)

where

\[
F = \frac{1}{2\pi i} \left\{ \begin{array}{l}
< \ln(\zeta_a - \tilde{\zeta}_a) > A^T + \sum_{k=1}^{4} < \ln(\zeta_a - \tilde{\zeta}_k) > B^T \bar{B} \bar{k} A^T, \\
F_{31} &= \frac{1}{2\pi i} (z_a - \tilde{z}_a) \ln(\zeta_a - \tilde{\zeta}_a) > A^T, \\
F_{32} &= \frac{1}{2\pi i} \sum_{k=1}^{4} (\zeta_a - \tilde{\zeta}_k) 1 - \bar{B}\bar{k} \bar{A} \ln(\zeta_a - \tilde{\zeta}_a) > B^T \bar{B} \bar{k} < \bar{a} > \bar{A}^T, \\
F_{33} &= \frac{1}{2\pi i} < c_a (\ln c_a - 1)(\zeta_a - \tilde{\zeta}_a) > A^T, \\
F_{34} &= \frac{1}{2\pi i} \sum_{k=1}^{4} (\zeta_a - \tilde{\zeta}_k) > B^T \bar{B} \bar{k} < c_a (\ln c_a - 1) > \bar{A}^T, \\
F_{35} &= \frac{-1}{2\pi i} < c_a \gamma_a \ln(-\tilde{z}_a)(\zeta_a - \tilde{\zeta}_a) > A^T, \\
F_{36} &= \frac{-1}{2\pi i} \sum_{k=1}^{4} (\zeta_a - \tilde{\zeta}_k) > B^T \bar{B} \bar{k} < c_a \gamma_a \ln(-\tilde{z}_a) > \bar{A}^T.
\end{array} \right\}
\] (7b)

In the above, \( I_k \) is a diagonal matrix whose the \( kk \) component is unity and all the others are zero; the overbar denotes the complex conjugate,

\[
\zeta_a = z_a + \frac{\sqrt{2^2 - \mu_a^2 b^2}}{a - i\mu_a b}, \quad \tilde{\zeta}_a = \tilde{z}_a + \frac{\sqrt{2^2 - \mu_a^2 b^2}}{a - i\mu_a b}, \quad c_a = \frac{1}{2} (a - ib \mu_a), \quad \gamma_a = \frac{a + ib \mu_a}{a - ib \mu_a}, \] (7d)

in which \( 2a \) and \( 2b \) are the major and minor axes of the elliptical hole. An elliptic hole can be made into a crack of length \( 2a \) by letting the minor axis \( 2b \) equal to zero. The fundamental solutions for crack problems can therefore be obtained from (7) by letting \( b = 0 \).

(iii) Laminates with elastic inclusions

Similar to the previous two cases, with the Green’s functions obtained in [7] the fundamental solutions for laminates with elastic inclusions can also be expressed by (5a-d) in which \( A \) and \( B \) are the material eigenvector matrices of the matrix, and \( f_i(\xi, x), i = 1, 2, 3, 4 \), are as follows.
\begin{align}
    f_i(\xi, x) &= [F_i(\zeta) + F_\varepsilon(\zeta)]i_1, \\
    f_\varepsilon(\xi, x) &= [F_j(\zeta) + F_\varepsilon(\zeta)]i_2, \\
    f_j(\xi, x) &= F_\varepsilon(\zeta)i_3, \\
    f_\zeta(\xi, x) &= [F_\varepsilon(\zeta) + F_\varepsilon(\zeta)]n_\xi(\zeta),
\end{align}

in which the detailed expressions of $F_i$, $F_\varepsilon$ and $F_j$ can be found in [8].

**Summary**

With the boundary integral equations and the explicit closed-form fundamental solutions provided in the last two sections, the boundary element for the coupled stretching-bending analysis of composite laminates can then be established and coded by following the steps described in the standard textbook such as [1].

**References**


