ABSTRACT: The problems of an infinite composite laminate with or without holes/cracks/inclusions subjected to concentrated forces and moments at any arbitrary points are considered in this paper. The elasticity solution of this problem can be used as a fundamental solution of boundary element method and is generally called Green’s function. The importance of Green’s function in constructing solutions to boundary value problems has been well recognized. Therefore, many analytical closed form solutions of Green’s functions have been obtained for several different problems such as two-dimensional infinite spaces, half-spaces, bimaterials, and the infinite spaces with the presence of cracks, holes or inclusions, etc. However, for those bending-stretching coupling problems, very few analytical Green’s functions have been obtained in the literature. For the unsymmetric composite laminates, the bending/twisting deformation will occur under pure inplane loadings, or the midplane stretching/shortening will occur under pure bending moments. To study this kind of problems, the recently developed complex variable formalism for coupled stretching-bending analysis called Stroh-like formalism is employed. By this formalism, the Green’s functions for the infinite unsymmetric composite laminates were obtained in closed-form. Moreover, based upon the non-defect Green’s functions, through the use of analytical continuation method the Green’s functions for holes/cracks/inclusions have also been obtained in explicit closed-form. In order to improve the current numerical methods for coupled stretching-bending analysis of the problems of holes/cracks/inclusions, here we like to recollect these solutions and use them to establish a powerful boundary element for general users.

KEYWORDS: Green’s function, Unsymmetric composite laminates, Holes, Cracks, Inclusions, Stretching-bending coupling

1. INTRODUCTION

For the Green’s function of the infinite laminates with bending extension coupling, detailed discussion and solutions have been provided in (Becker, 1995) by the complex variable formulation (Becker, 1991). However, due to the mathematical complexity Becker’s solution left a system of eight linear equations to be solved by numerical algorithm. This is
inconvenient when employing the Green’s function as a fundamental solution of the boundary element formulation to solve more practical engineering problems. To improve Becker’s solution, Hwu (2004) found an explicit analytical closed form Green’s function through the use of Stroh-like formalism for stretching-bending coupling analysis (Lu and Mahrenholtz, 1994; Cheng and Reddy, 2002; Hwu, 2003; Yin, 2003a,b).

As to the Green’s functions of holes/cracks in infinite laminates, most of the Green’s functions presented in the literature are for two-dimensional problems or pure bending problems (Lekhnitskii, 1968; Ting, 1996), and very few Green’s functions have been found in closed-form for the coupled stretching-bending analysis. Based upon the non-hole Green’s functions and through the use of analytical continuation method, recently the Green’s functions for holes in unsymmetric laminates are obtained in explicit closed-form for the complete loading cases (Hwu, 2005). The Green’s functions for cracks are then obtained by letting the minor axis of ellipse be zero (Hwu, 2006).

By following the steps of non-hole, hole and crack problems, the Green’s functions for elastic inclusions in composite laminates are also obtained recently by employing Stroh-like formalism (Hwu and Tan, 2007). The generality of the obtained Green’s functions can be shown as follows: (1) The laminates include any kinds of laminate lay-ups, symmetric or unsymmetric, which allow the stretching and bending deformations couple each other. (2) The concentrated forces and moments can be applied in in-plane and/or out-of-plane directions, located inside and/or outside the inclusions. (3) The elliptical elastic inclusions can be any kinds of elastic materials including the limiting cases such as holes, rigid inclusions, cracks, line inclusions, etc. The Green’s functions for inclusions are so general that they may cover all the previous solutions found for the non-hole and hole/crack problems.

It should be noted that all the Green’s functions for holes/cracks/inclusions in infinite laminates with stretching-bending coupling subjected to the in-plane concentrated forces and out-of-plane concentrated moments have exactly the same mathematical form as those of the corresponding two-dimensional problems, in which the only difference is the contents of the symbols. While for the other loading cases such as inplane torsions and transverse forces which play important roles in the laminate plate theory, their solution forms are different from those of the corresponding two-dimensional problems (Hwu, 2004; 2005; Hwu and Tan, 2007).

2. STROH-LIKE FORMALISM

Basic Equations

To describe the overall properties and macromechanical behavior of a composite laminate, the most popular way is the classical lamination theory (Jones, 1974). According to the observation of actual mechanical behavior of laminates, the Kirchhoff’s assumptions are usually made for the displacement fields, i.e., the laminate displacements $U_1$, $U_2$ and $U_3$ in the $x_1, x_2$ and $x_3$ directions are assumed to be

\[
U_i(x_1, x_2, x_3) = u_i(x_1, x_2) + x_3 \beta_i(x_1, x_2), \quad i = 1, 2, \quad \beta_1 = -w_{,1}, \quad \beta_2 = -w_{,2}.
\]

where

\[
U_3(x_1, x_2, x_3) = w(x_1, x_2),
\]

(2.1a)

(2.1b)
$u_1, u_2$ and $w$ are the middle surface displacements, and $\beta_i, i = 1,2$, are the negative of the slope of the middle surface in the $x_1$ and $x_2$ directions. The subscript comma stands for differentiation.

Based upon this assumption, the kinematic relations for small deformation, the constitutive laws for composite laminates, and the equilibrium equations for static conditions can be written in tensor notation as

$$
\begin{align*}
\varepsilon_{ij} &= \frac{1}{2} (u_{i,j} + u_{j,i}), \\
\kappa_{ij} &= \frac{1}{2} (\beta_{i,j} + \beta_{j,i}), \\
N_{ij} &= A_{ijkl} \varepsilon_{kl} + B_{ijkl} \kappa_{kl}, \\
M_{ij} &= B_{ijkl} \varepsilon_{kl} + D_{ijkl} \kappa_{kl}, \\
N_{ij,j} &= 0, \\
M_{ij,j} + q &= 0, \\
Q_i &= M_{ij,j}, \\
&\text{where } i, j, k, l = 1,2,
\end{align*}
$$

(2.2)

where $\varepsilon_{ij}$ and $\kappa_{ij}$ denote the mid-plane strain and plate curvature; $N_{ij}, M_{ij}$ and $Q_i$ denote the resultant forces, bending moments and shear forces; $A_{ijkl}, B_{ijkl}$ and $D_{ijkl}$ are, respectively, the extensional, coupling and bending stiffness tensors; $q$ is the lateral distributed load applied on the laminates. Repeated indices imply summation. Since in the classical lamination theory the basic equations (2.1)-(2.2) are seldom expressed in terms of the tensor notation, for those who want to have a clear picture of these equations please refer to (Hwu, 2003).

**General Solutions**

A general solution satisfying all the basic equations stated in (2.2) has been obtained and purposely arranged in the form of Stroh formalism of anisotropic elasticity, and hence is called Stroh-like formalism (Hwu, 2003). With this formalism, the solution fields of displacements and stresses are expressed as

$$
\begin{align*}
\mathbf{u}_d &= 2 \text{ Re}\{\mathbf{Af}(z)\}, \\
\phi_d &= 2 \text{ Re}\{\mathbf{Bf}(z)\},
\end{align*}
$$

(2.3a)

where

$$
\mathbf{u}_d = \begin{bmatrix} \mathbf{u} \\ \mathbf{\beta} \end{bmatrix}, \\
\phi_d = \begin{bmatrix} \phi \\ \psi \end{bmatrix}, \\
\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \\
\mathbf{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \\
\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \\
\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \\
\end{align*}
$$

(2.3b)

$$
\begin{align*}
\mathbf{f}(z) &= \begin{bmatrix} f_1(z_1) \\ f_2(z_2) \\ f_3(z_3) \\ f_4(z_4) \end{bmatrix}, \\
z_k &= x_1 + \mu_k x_2, \\
&\text{where } k = 1,2,3,4,
\end{align*}
$$

(2.3c)

$A = [a_1 \ a_2 \ a_3 \ a_4]$, $B = [b_1 \ b_2 \ b_3 \ b_4]$.

$\text{Re}$ stands for the real part of a complex number. In (2.3b), $\phi_1, \phi_2$ and $\psi_1, \psi_2$ are the stress functions related to the resultant forces $N_{ij}$, shear forces $Q_i$, effective shear forces $V_i$ and bending moments $M_{ij}$ by
where $\lambda_{ij}$ is the permutation tensor defined as
\[
\lambda_{11} = \lambda_{22} = 0, \quad \lambda_{12} = -\lambda_{21} = 1. \tag{2.4b}
\]
$f_k(z_k), k = 1,2,3,4$, are four holomorphic functions of complex variables $z_k$, which will be determined by the boundary conditions set for each particular problem. $\mu_i$ and $(a_k, b_k)$ are, respectively, the material eigenvalues and eigenvectors, which can be determined by the following eigen-relation
\[
N \xi = \mu \xi, \tag{2.5a}
\]
where $N$ is a $8 \times 8$ real matrix and $\xi$ is a $8 \times 1$ column vector defined by
\[
N = \begin{bmatrix}
N_1 & N_2 \\
N_3 & N_3^T
\end{bmatrix}, \quad \xi = \begin{bmatrix} a \\ b \end{bmatrix}. \tag{2.5b}
\]
The superscript $T$ denotes the transpose of a matrix. Detailed definitions of the submatrices $N_1, N_2$ and $N_3$ have been given in (Hwu, 2003). Moreover, the explicit expressions of $N_1, N_2$ and $N_3$ as well as their associated eigenvectors $a$ and $b$ have been found in (Hwu, 2003, Hsieh and Hwu, 2003).

By using the relations given in (2.4) and following the steps described in (Hsieh and Hwu, 2002, Hwu, 2003), the stress resultants $N_n, N_s, N_{ns}$, bending moments $M_n, M_s, M_{ns}$, shear forces $Q_n, Q_s$ and effective shear forces $V_n, V_s$ in the tangential-normal ($s$-$n$) coordinate system, can be obtained directly from the stress functions as
\[
N_n = n^T \phi_s, \quad N_{ns} = s^T \phi_s = -n^T \phi_a, \quad N_s = -s^T \phi_a, \\
M_n = n^T \psi_s, \quad M_{ns} = s^T \psi_s - \eta = -n^T \psi_{s,n} + \eta, \quad M_s = -s^T \psi_{s,n}, \\
Q_n = \eta_s, \quad Q_s = -\eta_s, \quad V_n = (s^T \psi_{s,n})_s, \quad V_s = -(n^T \psi_{s,n})_n,
\tag{2.6a}
\]
where
\[
\eta = \frac{1}{2} (s^T \psi_{s,n} + n^T \psi_{s,n}), \tag{2.6b}
\]
and
\[
s^T = (\cos \theta, \sin \theta), \quad n^T = (-\sin \theta, \cos \theta). \tag{2.6c}
\]

**Boundary Conditions**

Because in Ströh-like formalism the solution fields are expressed in terms of the generalized displacement and stress function vectors, $u_\phi$ and $\phi_\phi$, in order to employ this formalism to a
practical engineering problem we must transform the related displacement and force boundary conditions into the form of $u_d$ and $\phi_d$. From the displacement fields $U_i$ assumed in (2.1) and the generalized displacement vector $u_d$ defined in (2.3b) we see that the displacement boundary values usually described by using $u, v, w$ and $\beta_1, \beta_2$ have direct relation with $u_d$, and hence no further discussion about the displacement is needed. Moreover, through the relations given in (2.6) it is also very clear to describe the distributed force and moment boundary conditions in terms of $\phi_d$. For example, a traction free boundary condition usually written by

$$N_n = N_{as} = M_n = V_n = 0,$$

(2.7a)

can now be written in terms of the generalized stress function vector $\phi_d$ as

$$n^T \phi_s = s^T \phi_s = n^T \psi_s = (s^T \psi_s)_s = 0,$$

(2.7b)

or

$$\phi_s = 0.$$  
(2.7c)

Similarly, the displacements and surface tractions continuity conditions usually written by

$$u_n^{(1)} = u_n^{(2)}, \quad u_s^{(1)} = u_s^{(2)}, \quad \beta_n^{(1)} = \beta_n^{(2)}, \quad w_n^{(1)} = w_n^{(2)},$$

$$N_n^{(1)} = N_n^{(2)}, \quad N_n^{(1)} = N_n^{(2)}, \quad M_n^{(1)} = M_n^{(2)}, \quad V_n^{(1)} = V_n^{(2)},$$

along the interface,

(2.8a)

can now be rewritten as

$$u_n^{(1)} = u_n^{(2)}, \quad \phi_n^{(1)} = \phi_n^{(2)},$$

along the interface.  
(2.8b)

For a laminate subjected to concentrated forces and moments, we need to know how to describe the resultant forces $\tilde{t}_i$ and moments $\tilde{m}_i$ in terms of $\phi_d$, in which the resultant forces/moments are defined as the total forces/moments on a surface bounded by two arcs from A to B (lie on the top and bottom surfaces of the laminates) and two straight lines (normal to the laminate surface). Use of Cauchy's formula (Sokolnikoff, 1956) and the relations (2.4) for the stress resultants across the laminate thickness, simple relations have been obtained as follows (Hwu, 2004).

$$\tilde{t}_1 = \phi_1^{[p]} t_1, \quad \tilde{t}_2 = \phi_2^{[p]} t_1, \quad \tilde{t}_3 = \eta^{[p]} t_1,$$

$$\tilde{m}_1 = -(\psi_2 - x_2 \eta) t_1^{[p]}, \quad \tilde{m}_2 = (\psi_1 - x_1 \eta) t_1^{[p]},$$

$$\tilde{m}_3 = \int d x_1 d \phi_2 - x_2 d \phi_1 = (x_1 \phi_2 - x_2 \phi_1 - \Phi) t_1^{[p]},$$

(2.9a)

where

$$\phi_1 = -\Phi_{2,} \quad \phi_2 = \Phi_{1.}$$

(2.9b)

Note that the introduction of $\Phi$ is based upon the relation given in (2.4a)$_{1,2}$ that

$$N_{12} = \phi_{1,3} = N_{21} = -\phi_{2,2},$$

which is usually called Airy stress function.

By using the relations (2.9), the boundary conditions for concentrated force $\hat{f} = (\hat{f}_1, \hat{f}_2, \hat{f}_3)$ and moment $\hat{m} = (\hat{m}_1, \hat{m}_2, \hat{m}_3)$ at point $x = (\hat{x}_1, \hat{x}_2)$ can then be written as
\[ \oint_C d\phi = \hat{f}_1, \quad \oint_C d\phi_2 = \hat{f}_2, \quad \oint_C d\eta = \hat{f}_3, \]
\[ \oint_C d\psi_1 = \hat{m}_2 + (x_1 - \hat{x}_1)\hat{f}_2, \quad \oint_C d\psi_2 = -\hat{m}_1 + (x_2 - \hat{x}_2)\hat{f}_3, \]
\[ \oint_C d[(x_1 - \hat{x}_1)\phi_2 - (x_2 - \hat{x}_2)\phi_1 - \Phi] = \hat{m}_1, \]

(2.10)

where \( C \) is any closed contour enclosing the point \( \hat{x} \). Consideration of the force and moment equilibrium (2.10) and the single-valued requirement of displacements and slopes, the boundary conditions stating the concentrated forces and moments can then be expressed in terms of the generalized displacement and stress function vectors, \( \mathbf{u}_d \) and \( \mathbf{\phi}_d \) as follows (Hwu, 2004).

Case 1. \( \hat{f}_1, \hat{f}_2, \hat{m}_1, \hat{m}_2 \): \[ \oint_C d\mathbf{\phi}_d = \hat{\mathbf{p}}, \quad \oint_C d\mathbf{u}_d = \mathbf{0}, \quad \text{around the point } \hat{x}. \]

(2.11a)

Case 2. \( \hat{f}_3 \): \[ \oint_C d\eta = \hat{f}_3, \quad \oint_C d\mathbf{u}_{d,1} = \mathbf{0}, \quad \oint_C d\mathbf{u}_{d,2} = \mathbf{0}, \quad \text{around the point } \hat{x}. \]

(2.11b)

Case 3. \( \hat{m}_1 \): \[ \oint_C d\Phi = -\hat{m}_1, \quad \oint_C d\mathbf{w} = \mathbf{0}, \quad \text{around the point } \hat{x}, \]
\[ \mathbf{n}^T \hat{\mathbf{\phi}} = \mathbf{n}^T \hat{\mathbf{\psi}} = \mathbf{s}^T \hat{\mathbf{\psi}}_{,x} = \mathbf{0}, \quad \text{along any arbitrary surface boundary.} \]

(2.11c)

where
\[ \hat{\mathbf{p}} = (\hat{f}_1, \hat{f}_2, \hat{m}_2 - \hat{m}_1)^T. \]

GREEN’S FUNCTION FOR INFINITE LAMINATES

Consider an infinite laminate subjected to a concentrated force \( \mathbf{f} = (\hat{f}_1, \hat{f}_2, \hat{f}_3) \) and moment \( \mathbf{m} = (\hat{m}_1, \hat{m}_2, \hat{m}_3) \) at point \( \hat{x} = (\hat{x}_1, \hat{x}_2) \). The elasticity solution of this problem can be used as a fundamental solution of boundary element method and is generally called Green’s function. The boundary conditions stating the concentrated forces and moments have been shown in (2.11). To find a solution satisfying the conditions (2.11), the choice of \( \mathbf{f}(z) \) in the general solutions (2.3) is very critical in the solution procedures. Equations (2.11) show that the stress functions \( \phi_1, \phi_2, \psi_1 \) and \( \psi_2 \) should be multi-valued functions if the concentrated forces \( \hat{f}_1, \hat{f}_2 \) and moments \( \hat{m}_1, \hat{m}_2 \) are applied, whereas \( \eta(=\psi_{x,k}/2) \) and \( \Phi(=\int \phi_2 dx_1) \) should be multi-valued functions if the concentrated forces \( \hat{f}_3 \) and moments \( \hat{m}_3 \) are applied. However, no matter which kind of force conditions are considered, the physical quantities such as the displacements and slopes should always be single-valued to confirm that the laminates will not break off when deformed. Since \( \phi_1 \) (or \( \psi_1 \)), \( \psi_{x,k} \) and \( \int \phi_2 dx_1 \) stand for three different function status, in the followings the force conditions are all separated into three cases, i.e., (1) \( \hat{f}_1, \hat{f}_2, \hat{m}_1, \hat{m}_2 \); (2) \( \hat{f}_3 \); (3) \( \hat{m}_3 \).

With proper selection of \( \mathbf{f}(z) \), through satisfaction of boundary conditions (2.11), the unknown complex function vector \( \mathbf{f}(z) \) of (2.3) has been determined to be (Hwu, 2004).
Case 1. \( f(z) = \frac{1}{2\pi i} \langle \log(z_a - \hat{z}_a) > A^T \hat{p}; \)  

\[ \text{(3.1a)} \]

Case 2. \( f(z) = \frac{\hat{f}_3}{2\pi i} \langle (z_a - \hat{z}_a) [\log(z_a - \hat{z}_a) - 1] > A^T i_3; \)  

\[ \text{(3.1b)} \]

Case 3. \( f(z) = \frac{\hat{m}_3}{2\pi i} \langle \frac{1}{z_a - \hat{z}_a} > A^T i_2. \)  

\[ \text{(3.1c)} \]

where \( i_2 \) and \( i_3 \) are two 4×1 unit column vectors with unit value in the 2\textsuperscript{nd} and 3\textsuperscript{rd} component, respectively, and the other components are zero. The angular bracket stands for the diagonal matrix whose components vary according to the subscript \( \alpha \), \( \alpha = 1,2,3,4 \), i.e., \( \langle f_\alpha \rangle = \text{diag}[f_1, f_2, f_3, f_4] \).

GREEN’S FUNCTION FOR HOLES/CRACKS IN LAMINATES

Consider an infinite composite laminate containing an elliptical hole under a concentrated force \( \hat{f} \) and moment \( \hat{m} \) at point \( \hat{x} \). The contour of the hole boundary is represented by

\[ x_1 = a \cos \omega, \quad x_2 = b \cos \omega, \]  

\[ (4.1) \]

where \( 2a, 2b \) are the major and minor axes of the ellipse and \( \omega \) is a real parameter related to the tangent angle \( \theta \) by

\[ \rho \cos \theta = -a \sin \omega, \quad \rho \sin \theta = b \cos \omega, \]  

\[ (4.2a) \]

where

\[ \rho^2 = a^2 \sin^2 \omega + b^2 \cos^2 \omega. \]  

\[ (4.2b) \]

If the hole is assumed to be traction free, the boundary conditions of this problem can be described by (2.7c) for the points along the hole boundary and by (2.11) for the concentrated forces/moments. Similar to the associated two-dimensional problems, this problem can be solved through the use of a transformation function which will map the elliptical hole boundary in the \( z \)-plane to four different slanted elliptical hole boundary in the \( k \)-plane, and then to the same unit circle in the \( \zeta \)-plane. The relation between \( z_a \) and \( \zeta_k \) is

\[ z_a = \frac{1}{2} \left\{ (a - ib\mu_a) \zeta_a + (a + ib\mu_a) \frac{1}{\zeta_a} \right\}, \quad \alpha = 1, 2, 3, 4, \]  

\[ (4.3a) \]

or inversely

\[ \zeta_a = \frac{z_a + \sqrt{z_a^2 - a^2 - b^2 \mu_a^2}}{a - ib\mu_a}, \quad \alpha = 1, 2, 3, 4. \]  

\[ (4.3b) \]

With the non-hole solution (3.1) as the basis, transforming the variable from \( z_a \) to \( \zeta_k \) according to the transformation function (4.3), and then employing the method of analytical continuation will give us the following solutions (Hwu, 2005)
Case 1: \( f(\zeta) = \frac{1}{2\pi i} \left\{ \log(\zeta'_{\alpha} - \zeta_{\alpha}) > A^{T} + \sum_{k=1}^{4} \log(\zeta_{\alpha}^{-1} - \zeta_{k}^{-1}) > B^{-1}B_{k}A^{T} \right\} \hat{p} ; \) (4.4a)

Case 2: \( f(\zeta) = (z_{\alpha} - \zeta_{\alpha}) \log(\zeta_{\alpha} - \zeta_{\alpha}) > \mathbf{q}_{2} \)
\[
- \sum_{k=1}^{4} \log(\zeta_{\alpha}^{-1} - \zeta_{k}) (1 - \frac{\gamma_{\alpha} \zeta_{\alpha}^{-1}}{\zeta_{k}}) \log(\zeta_{\alpha}^{-1} - \zeta_{k}) > B^{-1}B_{k}\mathbf{q}_{k}
\]
\[
+ < (\zeta_{\alpha} - \zeta_{\alpha}) > \mathbf{q}_{c}^{*} - \sum_{k=1}^{4} < (\zeta_{\alpha}^{-1} - \zeta_{k}) > B^{-1}B_{k}\mathbf{q}_{k}^{*}
\]
\[
- < (\zeta_{\alpha}^{-1} - \zeta_{\alpha}^{-1}) > \mathbf{q}_{c}^{**} + \sum_{k=1}^{4} < (\zeta_{\alpha}^{-1} - \zeta_{k}^{-1}) > B^{-1}B_{k}\mathbf{q}_{k}^{**} .
\]
(4.4b)

Case 3: \( f(\zeta) = \frac{1}{\zeta_{\alpha} - \zeta_{\alpha}} > \mathbf{q}_{3}^{*} - \sum_{k=1}^{4} \frac{1}{\zeta_{\alpha}^{-1} - \zeta_{k}^{-1}} > B^{-1}B_{k}\mathbf{q}_{3}^{**} .
\)
(4.4c)

in which
\[
\mathbf{q}_{c} = c_{\alpha} > \mathbf{q}_{2}, \quad \mathbf{q}_{c}^{*} = c_{2\alpha} > \mathbf{q}_{2}, \quad \mathbf{q}_{c}^{**} = c_{3\alpha} > \mathbf{q}_{2},
\]
\[
\mathbf{q}_{3}^{*} = c_{4\alpha} \hat{\zeta}_{\alpha} > \mathbf{q}_{3}, \quad \mathbf{q}_{3}^{**} = c_{4\alpha} \gamma_{\alpha} / \hat{\zeta}_{\alpha} > \mathbf{q}_{3},
\]
(4.5a)

and
\[
\gamma_{\alpha} = \frac{a + ib\mu_{\alpha}}{a - ib\mu_{\alpha}}, \quad c_{\alpha} = \frac{1}{2} (a - ib\mu_{\alpha}), \quad c_{2\alpha} = c_{\alpha} (\log c_{\alpha} - 1),
\]
\[
c_{3\alpha} = c_{\alpha} \gamma_{\alpha} \log(-\hat{\zeta}_{\alpha}), \quad c_{4\alpha} = [c_{\alpha} (\hat{\zeta}_{\alpha} - \gamma_{\alpha} / \hat{\zeta}_{\alpha})]^{-1},
\]
(4.5b)

As to the Green’s functions for laminates containing cracks, solutions can be obtained from (4.4) and (4.5) by letting the minor axis 2b of the ellipse be equal to zero. The final simplified solutions and related discussions can be found in (Hwu, 2006).

GREEN’S FUNCTION FOR ELASTIC INCLUSIONS IN LAMINATES

Consider an infinite composite laminate containing an elliptical inclusion subjected to a concentrated force \( \hat{\mathbf{f}} \) and moment \( \hat{\mathbf{m}} \) at point \( \mathbf{x} \). The contour of the inclusion boundary is represented by (4.1). The inclusion and the matrix (the laminates) are assumed to be perfectly bonded along the interface, and hence the displacements and surface tractions across the interface should be continuous. The continuity condition can be expressed by (2.8b).

Similar to the hole problems, to find a suitable function satisfying the boundary conditions around the elliptical inclusion the transformation function (4.3) is a proper choice for the function expressions. Different from the hole problems, we should now note that although the transformation (4.3) will map the points outside the elliptic inclusion into the points outside the unit circle in \( \zeta_{\alpha} \)-domain, the points inside the elliptic inclusion will be mapped to two
different points in $\zeta^{\alpha}$-domain and is a one-to-one transformation only when the following restriction is satisfied (Hwu and Yen, 1993)

$$f(\sqrt{\gamma^{\alpha}}e^{i\alpha}) = f(\sqrt{\gamma^{\alpha}}e^{-i\alpha}).$$  \hspace{1cm} (5.1)

and its mapped field is an annular ring of $\sqrt{m_{\alpha}} \leq \zeta^{\alpha} \leq 1$, where $m_{\alpha}$ is the absolute value of $\gamma^{\alpha}$. Because the mapped conditions are different for the points inside or outside the inclusions, the solutions should then be separated into two different cases. One is forces/moments outside the inclusions, and the other is forces/moments inside the inclusions. Again, the method of analytical continuation can be employed to solve this problem. To make this method successful, the function $f(z)$ is separated into three parts: singular, holomorphic inside the unit circle, and holomorphic outside the unit circle. With this understanding, by following the standard approach of analytical continuation method the solutions have been obtained as (Hwu and Tan, 2007)

**Forces/moments outside the inclusions**

$$\mathbf{u}^{(1)}_d = 2 \text{Re}\{\mathbf{A}_1[f_0(\zeta) + \tilde{f}_1(\zeta)]\}, \quad \phi^{(1)}_d = 2 \text{Re}\{\mathbf{B}_1[f_0(\zeta) + \tilde{f}_1(\zeta)]\},$$

$$\mathbf{u}^{(2)}_d = 2 \text{Re}\{\mathbf{A}_2 f_2(\zeta^*)\}, \quad \phi^{(2)}_d = 2 \text{Re}\{\mathbf{B}_2 f_2(\zeta^*)\},$$ \hspace{1cm} (5.2)

Case 1: $f_0^-(\zeta) = \frac{1}{2\pi i} \text{log}(\zeta^a - \tilde{\zeta}^a) > \mathbf{A}_1^T \hat{\mathbf{p}}$

$$\tilde{f}_1(\zeta) = -\sum_{k=1}^{\infty} \zeta^{-k} \mathbf{A}_1^{-1}\{\mathbf{A}_1 \mathbf{e}^* - \mathbf{A}_2 \zeta^{-k} \}

\quad \text{or} \quad \tilde{f}_1(\zeta) = -\sum_{k=1}^{\infty} \zeta^{-k} \mathbf{B}_1^{-1}\{\mathbf{B}_1 \mathbf{e}^* - \mathbf{B}_2 \zeta^{-k} \}

f_2(\zeta^*) = \sum_{k=\infty}^{\infty} \zeta^{+k} \mathbf{e}_k, \quad \mathbf{e}_{-k} = \gamma^{-k} \mathbf{e}_k

\mathbf{e}^{-k} = \frac{1}{2\pi i} \left< \frac{-1}{k} \zeta^{-k} > \mathbf{A}_1^T \hat{\mathbf{p}} \right> \hspace{1cm} (5.3a)

Case 2: $f_0^-(\zeta) = \frac{\hat{f}_3}{2\pi i} < (z - \tilde{z}) \text{log}(\zeta^a - \tilde{\zeta}^a) + c_{2\alpha}(\zeta^a - \tilde{\zeta}^a) + c_{3\alpha}(\zeta^{-1} - \tilde{\zeta}^{-1}) > \mathbf{A}_1^T \mathbf{i}_3$

$$\tilde{f}_1(\zeta), f_3(\zeta^*) : \text{same mathematical form as Case 1, the only difference is the content of } \mathbf{e}^{-k} \text{ and } \mathbf{e}_k$$

$$\mathbf{e}^{-k} = \frac{\hat{f}_3}{2\pi i} \left< \frac{c_{2a}(\tilde{\zeta}^{k} - \tilde{\zeta}^{-k}) + \gamma^{-k}}{2\zeta^{a}} > \mathbf{A}_1^T \mathbf{i}_3, \quad k \neq 1 \hspace{1cm} (5.3b)$$

$$\mathbf{e}^{-k} = \frac{\hat{f}_3}{2\pi i} \left< \frac{-c_{2a}}{k\zeta^{k}} \frac{k-1}{k+1} > \mathbf{A}_1^T \mathbf{i}_3, \quad k \neq 1$$
Case 3: $f_0^-(\zeta) = \frac{\hat{m}_3}{\sqrt{2m}} < \frac{c_4 \hat{z}_a}{\zeta_a - \hat{z}_a} > A_1^T i_2$

$\mathbf{f}_1(\zeta), \mathbf{f}_2(\zeta^*)$: same mathematical form as Case 1, the only difference is the content of $e_k^-$ and $c_k$

$$e_k^- = -\frac{\hat{m}_3}{\sqrt{2m}} < c_4 \hat{z}_a > A_1^T i_2$$

(5.3c)

in which

$$e_k = \{G_0 - \overline{G}_k \overline{G}_0^{-1} G_k \}^{-1} \{t_k - \overline{G}_k \overline{G}_0^{-1} \mathbf{i}_k\}, k = 1, 2, \cdots, \infty,$$

(5.4a)

$$G_0 = \{\overline{M}_1 + M_2\}A_2, \quad G_k = \{M_1 - M_2\}A_2 < \gamma_{ak} >, \quad t_k = -iA_1^{-T} e_k^-,$$

(5.4b)

$$M_1 = -iB_1A_1^{-1}, \quad M_2 = -iB_2A_2^{-1}.$$  

(5.4c)

**Forces/moments inside the inclusions**

$$u_d^{(i)} = 2 \text{Re} \{A_i[f_0(\zeta) + f_1(\zeta)]\}, \quad \phi_d^{(i)} = 2 \text{Re} \{B_i[f_0(\zeta) + f_1(\zeta)]\},$$

$$u_d^{(2)} = 2 \text{Re} \{A_2[f_0^*(\zeta^*) + f_2(\zeta^*)]\}, \quad \phi_d^{(2)} = 2 \text{Re} \{B_2[f_0^*(\zeta^*) + f_2(\zeta^*)]\},$$

(5.5)

**Case 1:** $f_0(\zeta) = \frac{1}{2\sqrt{m}} < \log \zeta_a > A_1^T \mathbf{p}$

$$f_0^*(\zeta^*) = \frac{1}{2\sqrt{m}} < \log(z_a^* - \hat{z}_a^*) > A_2^T \mathbf{p}$$

$$f_1(\zeta) = \sum_{k=1}^{\infty} < \zeta_a^{-k} > A_1^{-1} \{A_2 e_k^+ + A_2 < \gamma_{ak} > c_k + \overline{A}_2 \overline{c}_k\},$$

or

$$f_1(\zeta) = \sum_{k=1}^{\infty} < \zeta_a^{-k} > B_1^{-1} \{B_2 e_k^+ + B_2 < \gamma_{ak} > c_k + \overline{B}_2 \overline{c}_k\}$$

$$f_2^*(\zeta^*) = \sum_{k=\infty}^{\infty} < \zeta_a^{-k} > e_k, \quad e_{-k} = < \gamma_{ak} > e_k$$

$$e_k^+ = -\frac{1}{2\sqrt{m}i} < \hat{z}_a^* + (\gamma_{ak}^* / \zeta_a^*)^k > A_2^T \mathbf{p},$$

(5.6a)

**Case 2:** $f_0(\zeta) = < \log \zeta_a > \{< \zeta_a > d_1 + < \zeta_a^{-1} > d_{-1} + d_0 \} + < \zeta_a^{-1} > k_{-1} + < \zeta_a > k_1$

$$f_0^*(\zeta^*) = \frac{\hat{f}_1}{2\sqrt{m}} < (z_a^* - \hat{z}_a^*)[\log(z_a^* - \hat{z}_a^*) - 1] > A_2^T i_3$$

$f_1(\zeta), f_2(\zeta^*)$: same mathematical form as Case 1, the only difference is the content of $e_k^+$ and $c_k$. 
\[ e_i = \frac{\hat{f}_1}{2\pi} < c_i^* \left( 1 \left[ \hat{z}_{\alpha}^{x^2} + \left( \gamma_{\alpha}^{x^*} / \zeta_{\alpha}^{x^*} \right)^k \right] \right) > A^*_i, \]  
\[ e_k = \frac{\hat{f}_1}{2\pi} \frac{c_k^*}{k} \left( 1 \left[ \hat{z}_{\alpha}^{x^2} + \left( \gamma_{\alpha}^{x^*} / \zeta_{\alpha}^{x^*} \right)^k \right] \right) > A^*_k, \quad k \neq 1, \]  
\[ d_{1}, d_{-1}, d_{0}, k_{-1}, k_{1} : \text{eqns.} (6.6b), (6.7), (6.8a-e) \text{ of (Hwu and Tan, 2007)} \]

Case 3: \( f_0(\zeta) = 0 \)

\[ f_0^*(\zeta^*) = \frac{\hat{m}_1}{2\pi} < \frac{1}{\zeta^* - \hat{z}_{\alpha}^*} > A^*_i, \]

\[ f_1(\zeta), f_2(\zeta^*) : \text{same mathematical form as Case 1, the only difference is the content of } e_k, \text{ and } c_k \]

\[ e_k^* = \frac{\hat{m}_1}{2\pi} < \frac{\hat{z}_{\alpha}^{x^*} - \left( \gamma_{\alpha}^{x^*} / \zeta_{\alpha}^{x^*} \right)^k}{c_k^* \left[ \hat{z}_{\alpha}^{x^*} - \left( \gamma_{\alpha}^{x^*} / \zeta_{\alpha}^{x^*} \right)^k \right]} > A^*_k, \]  
\[ \text{in which } e_k \text{ is the same expression as that shown in (5.4a) except now} \]

\[ t_k = - (\overline{M}_1 - \overline{M}_2) \overline{A}_x \overline{e}_k^*. \]  

CONCLUSIONS

The Green’s functions for holes/cracks/inclusions in composite laminates with stretching-bending coupling have been obtained and presented in explicit closed-form. With these Green’s functions and the boundary integral equations for bending-stretching coupling analysis of composite laminates, it is expected that special boundary elements for holes/cracks/inclusions problems can be designed. The special feature of this boundary element will then be no meshes are needed around holes/cracks/inclusions because the conditions around holes/cracks/inclusions boundaries have been exactly satisfied. Therefore, an accurate and efficient boundary element for hole/cracks/inclusions of general coupling analysis is expected to be developed in our next research work.

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