Boundary Integral Equations for the Bending-Stretching Coupling Analysis of Composite Laminates

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ABSTRACT: According to the reciprocal theorem of Betti and Raleigh, the boundary integral equations for the bending-stretching coupling analysis of composite laminates are derived in this paper. To make the boundary integral equations work for the programming of boundary element codes, several fundamental solutions such as those for the infinite laminates, the laminates with holes/cracks, are presented in this paper. With the boundary integral equations and the explicit fundamental solutions, the boundary element formulation for the bending-stretching coupling analysis of composite laminates can then be established.

1 INTRODUCTION
Consider the bending-stretching coupling analysis of composite laminates, no associated boundary integral equations, which are the basis for the programming of boundary element codes, can be found in the literature. Only the boundary integral equations for the pure bending or pure stretching analysis have been presented (Brebbia et al., 1984). According to the reciprocal theorem of Betti and Raleigh (Sokolnikoff, 1956), the general boundary integral equations for the bending-stretching coupling analysis of composite laminates are derived in this paper.

To make the boundary integral equations work for the programming of boundary element codes, the next important task is the derivation of the fundamental solutions, which are the solutions to the problems subjected to concentrated forces and moments at arbitrary points. The importance of fundamental solutions in constructing boundary elements has been well recognized. Many analytical closed form fundamental solutions have been obtained for several different problems such as the two-dimensional infinite spaces, half-spaces, bimaterials, and the infinite spaces with the presence of cracks, holes or inclusions, etc. (Hwu and Yen, 1991; 1993; Ting, 1996). For the composite laminates with bending-stretching coupling, detailed discussion and solutions have been provided by using the complex variable formulation such as (Becker, 1995). However, due to mathematical complexity most of the solutions presented in the literature left a system of linear algebraic equations to be solved numerically. This is inconvenient when we employ the fundamental solutions to the boundary element formulation for solving more practical engineering problems. Recently, by employing Stroh-like formalism for the bending-stretching coupling analysis of composite laminates, several fundamental solutions have been obtained such as those for the infinite laminates (Hwu, 2004), the laminates with holes/cracks (Hwu, 2005), and the laminates with inclusions (Hwu and Tarn, 2006), etc. For the convenience of readers, some of these solutions will also be listed in this paper. With the boundary integral equations and the explicit closed form fundamental solutions, the boundary element formulation for the bending-stretching coupling analysis of composite laminates can then be established.

2 BOUNDARY INTEGRAL EQUATION
The reciprocal theorem of Betti and Raleigh in terms of stresses and strains can be expressed as (Sokolnikoff, 1956)

\[ \int_{\Omega} \sigma_{ij} \epsilon_{ij} \, d\Omega = \int_{\Omega} \sigma_{ij}^* \epsilon_{ij}^* \, d\Omega, \quad \]  

(1)

where \( \sigma_{ij}, \epsilon_{ij} \) and \( \sigma_{ij}^*, \epsilon_{ij}^* \), \( i,j = 1,2,3 \), are the stresses and strains induced by two different loading systems on the same elastic body whose region is denoted by \( \Omega \). If the elastic body is a moderately thick laminated plate, according to the Mindlin’s assumptions the integration of (1) with respect to the thickness leads to
\[ \int_A \left( N_x \varepsilon_{x0} + N_y \varepsilon_{y0} + N_{xy} \gamma_{xy0} + M_x \kappa_x^* + M_y \kappa_y^* + Q_x \gamma_{x0}^* + Q_y \gamma_{y0}^* \right) dA \]

(2)

where \((N_x, N_y, N_{xy}, (M_x, M_y, M_{xy}))\) and \((Q_x, Q_y)\) are the stress resultants, bending moments and transverse shear forces; \((\varepsilon_{x0}, \varepsilon_{y0}, \gamma_{xy0}), (\kappa_x, \kappa_y, \kappa_{xy})\) and \((\gamma_{xz}, \gamma_{yz})\) are the midplane strains, curvatures and transverse shear strains, which are related to the midplane displacements \(u_0, v_0\) and \(w_0\) by

\[ \varepsilon_{x0} = \frac{\partial u_0}{\partial x}, \quad \varepsilon_{y0} = \frac{\partial v_0}{\partial y}, \quad \gamma_{xy0} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}, \]

\[ \kappa_x = \frac{\partial \beta_x}{\partial x}, \quad \kappa_y = \frac{\partial \beta_y}{\partial y}, \quad \kappa_{xy} = \frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}, \]

\[ \beta_x = \gamma_{xz} - \frac{\partial w}{\partial x}, \quad \beta_y = \gamma_{yz} - \frac{\partial w}{\partial y}. \]

With the relations (3), the surface integral of (2) can be reduced to a line integral by integrating by parts, for example,

\[ \int_A N_x \varepsilon_{x0}^* dA = \int_A N_x \frac{\partial u_0^*}{\partial x} dA \]

(4)

\[ = \int_A N_x u_0^* n_x d\Gamma - \int_A \frac{\partial N_x}{\partial x} u_0^* dA, \ldots, \text{etc.,} \]

in which \(\Gamma\) is the boundary of area \(A\) whose normal direction is denoted by \((n_x, n_y)\). After integrating by parts term by term as shown in (4), equation (2) can be separated into two parts. One is surface integral and the other is line integral. By employing the equilibrium equations of the plates for the surface integral such as

\[ \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + q_x = 0, \ldots, \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0, \ldots, \]

\[ \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x, \ldots, \text{etc.,} \]

(5)

where \(q_x, q_y\) and \(q\) are body forces in \(x, y\) and \(z\) directions, equation (2) can be reduced to

\[ \int_A (q_x u_0^* + q_y v_0^* + q w^*) dA + \int_\Gamma \left[ (N_x u_0^* + N_{xy} v_0^* + N_y w_0^* + M_x \beta_x^* + M_y \beta_y^* + Q_x \gamma_{x0}^* + Q_y \gamma_{y0}^* \right] d\Gamma \]

(6)

By using the following transformation relations,

\[ u_0 = u_n n_x - u_{n_y}, \quad v_0 = u_n n_y + u_{n_x}, \]

\[ \beta_x = \beta_{n_x} - \beta_{n_y}, \quad \beta_y = \beta_{n_y} + \beta_{n_x}, \]

\[ Q_n = Q_n n_x + Q_n n_y, \]

(7)

\[ N_n = N_n n_x^2 + 2N_{nxy} n_x n_y + N_y n_y^2, \]

\[ N_{ns} = (N_y - N_x) n_x n_y + N_{xy} (n_x^2 - n_y^2), \]

\[ M_n = M_n n_x^2 + 2M_{nxy} n_x n_y + M_y n_y^2, \]

\[ M_{ns} = (M_y - M_x) n_x n_y + M_{xy} (n_x^2 - n_y^2), \]

where the subscripts \(n\) and \(s\) denote the values in normal and tangential directions, equation (6) can be further reduced to

\[ \int_A (q_x u_0^* + q_y v_0^* + q w^*) dA + \int_\Gamma \left[ (N_n u_0^* + N_{ns} u_0^* + M_n \beta_x^* + M_{ns} \beta_y^* + Q_n \gamma_{x0}^* \right] d\Gamma \]

(8)

For thin plates, Kirchhoff’s assumptions apply and transverse shear deformations are usually ignored, i.e.,

\[ \gamma_{xz} = \gamma_{yz} = 0. \]

With this assumption and the last two equations of (3), the rotation angles \(\beta_x\) and \(\beta_y\) will not be independent but be related to the deflection \(w\) by

\[ \beta_x = -\frac{\partial w}{\partial x}, \quad \beta_y = -\frac{\partial w}{\partial y}. \]
Using the transformation relations shown in (7)\textsubscript{3,4}, we have
\[ \beta_n = -\frac{\partial w}{\partial n}, \quad \beta_s = -\frac{\partial w}{\partial s}. \] (11)

Substituting (11) into (8) and integrating by parts as shown in (4), we get
\[ \int_A (q_x u^*_0 + q_y v^*_0 + q^*_w) dA + \]
\[ \int_{\Gamma} \left( N_n u^*_n + N_m u^*_m + M_n \beta^*_n + V_n w^* \right) d\Gamma - (M_w^*) \int_{\Gamma} = \]
\[ = \int_A (q_x u_0 + q_y v_0 + q^*_w) dA + \]
\[ \int_{\Gamma} \left( N_n u_n + N_m u_m + M_n \beta_n + V_n w \right) d\Gamma - (M_w^*) \int_{\Gamma} = \]
where \( V_n \) is the effective shear force defined as
\[ V_n = Q_n + \frac{\partial M_n^s}{\partial S}, \] (13)
and \( \Gamma^- \) and \( \Gamma^+ \) represent, respectively, the starting and final points of the boundary \( \Gamma \). If the \( M_w^* \) or \( M_n^s \) value is continuous, the last terms of both sides of (12) will vanish. Otherwise, the addition of these two terms becomes necessary, which may occur if the \( M_n^s \) value is discontinuous.

When the boundary has many corners, the last terms of both sides of (12) may be represented as
\[ (M_w^*)_{\Gamma^-} = \sum_{k=1}^{N_c} (M_{n,k}^s - M_{m,k}^s) w^*_k, \] \[ (M_n^s w^*)_{\Gamma^-} = \sum_{k=1}^{N_c} (M_{n,k}^s - M_{n,k}^s) w^*_k, \] (14)
where the subscript \( k \) stands for the value in the \( k \)th corner, the superscripts + and – denote, respectively, the value behind and ahead of the corner and \( N_c \) is the number of corners. (Figure 1)

Consider the body force \( \mathbf{q}^* = (q^*_x, q^*_y, q^*_z) \) to be a unit point load applied at the point \( \xi \) in each of three orthogonal directions given by the unit vector \( \mathbf{e}_i, \quad i = 1, 2, 3, \) i.e.,
\[ q^*_x(x) = \delta(\xi, x) \mathbf{e}_1, \quad \text{or} \quad q^*_y(x) = \delta(\xi, x) \mathbf{e}_2, \] (15)
\[ q^*_z(x) = \delta(\xi, x) \mathbf{e}_3, \]
where \( \delta(\xi, x) \) represents the Dirac delta function, \( \xi \) is the singular load point and \( \mathbf{x} \in A \) is the field point. Substituting each point load of (15) independently into (12) and using (14), we get
\[ u_i(\xi) + \int_{\Gamma} t^*_j(\xi, x) u_j(x) d\Gamma - \sum_{k=1}^{N_c} u^*_j(\xi, x) u_j(x) \]
\[ = \int_{\Gamma} u^*_j(\xi, x) t^*_j(x) d\Gamma + \int_A u^*_j(\xi, x) q^*_j(x) dA - \]
\[ \sum_{k=1}^{N_c} u^*_j(\xi, x) t^*_j(x), \quad i = 1, 2, 3, \quad j = 1, 2, 3, 4, \]
in which new notations are used for the convenience of later presentation. They are
\[ u_1 = u_x, \quad u_2 = u_y, \quad u_3 = u_z, \quad u_4 = \beta_n, \]
\[ t_1 = T_x n_x + N_{xy} n_y, \quad t_2 = T_y n_x + N_{xy} n_y, \]
\[ t_3 = V_n, \quad t_4 = M_n, \]
\[ q_1 = q_x, \quad q_2 = q_y, \quad q_3 = q_z, \quad q_4 = m_n = 0, \]
\[ t_c = M_n^c - M_n^s, \]
and \( u^*_j(\xi, x), \ t^*_j(\xi, x) \) and \( t^*_n(\xi, x) \) represent \( u_j, \ t_j \) and \( t_n \) at point \( x \) corresponding to a unit point force acting in the \( \mathbf{e}_i \) direction applied at point \( \xi \).

Note that in derivation of (16), the following relations have been used
\[ N_n u^*_n + N_m u^*_m = T_j u^*_0 + T_j v^*_0, \]
\[ N_n^s u^*_n + N_m^s u^*_m = T_j u^*_0 + T_j v^*_0. \] (18)

Since \( t^*_j(\xi, x) \) are the stress resultants corresponding to the unit point load applied at \( \xi \) directing in \( \mathbf{e}_i \), when \( x \) approaches to \( \xi \) they will become singular. Therefore, if \( \xi \) goes to the boundary \( \Gamma \), the integral on the right hand side of (16) should be taken in the sense of Cauchy principal value and equation (16) can be rewritten as
where \( c_y(\xi) \) is the coefficient containing the corresponding principal value which can be indirectly computed by applying (19) to represent rigid body movements.

Notice that we now have four unknown functions, i.e., \( u_j \) or \( t_j, j = 1,2,3,4 \). Hence, in addition to the three equations of (19), we need another equation to solve the problem. This equation is given by finding the derivative of the third equation of (19) with respect to the normal, which gives

\[
(c_{3j}(\xi) \frac{\partial u_j(\xi)}{\partial n} + \frac{\partial t_{ij}(\xi, x)}{\partial n} u_j(x)) d\Gamma - \sum_{k=1}^{N} \frac{\partial t_{ij}}{\partial n} u_j(x_k) = \int_{\Gamma} u_{ij}^*(\xi, x) t_j(x) d\Gamma + \int_{A} u_{ij}^*(\xi, x) q_j(x) dA - \sum_{k=1}^{N} u_{ij}^*(\xi, x_k) t_j(x_k), \quad i = 1,2,3, \quad j = 1,2,3,4, \tag{20}
\]

The four equations provided in (19) and (20), called the boundary integral equations, then constitute the basis of the boundary element formulation. To make these equations work for the programming of boundary element codes, we need to find the fundamental solutions related to \( t_{ij}^*(\xi, x), \ u_{ij}^*(\xi, x) \ and \ t_{ij}^* \) given in (19) and (20).

3 FUNDAMENTAL SOLUTIONS

Recently, by employing Stroh-like formalism for the bending-stretching coupling analysis of composite laminates, several fundamental solutions have been obtained such as those for the infinite laminates (Hwu, 2004), the laminates with holes/cracks (Hwu, 2005), and the laminates with inclusions (Hwu and Tarn, 2006), etc. The solutions obtained by using Stroh-like formalism are usually expressed in complex matrix form as (Hwu, 2003)

\[
u_a = 2 \text{Re}\{Af(z)\}, \quad \phi_a = 2 \text{Re}\{Bf(z)\}, \tag{21}
\]

where \( \text{Re} \) denotes real part of a complex number; \( \mathbf{u}_a \) and \( \mathbf{\phi}_a \) are, respectively, the generalized displacement and stress function vectors; \( \mathbf{A} \) and \( \mathbf{B} \) are two 4×4 material eigenvector matrices; \( \mathbf{f}(z) \) is a 4×1 complex function vector to be determined through the satisfaction of boundary conditions. Detailed explanation of the above symbols can be found in (Hwu, 2003). Followings are the solutions of \( \mathbf{f}(z) \) for the composite laminates with or without holes/cracks.

3.1 Infinite laminates (Hwu, 2004)

Case 1: \( \hat{\mathbf{f}}_1, \hat{\mathbf{f}}_2, \hat{\mathbf{m}}_1, \hat{\mathbf{m}}_2 \)

\[
\mathbf{f}(z) = \frac{1}{2\pi i} \text{Re}(z_a - \bar{z}_a) > \mathbf{A}^T \hat{\mathbf{p}}. \tag{22}
\]

Case 2: \( \hat{\mathbf{f}}_3 \)

\[
\mathbf{f}(z) = \frac{1}{2\pi i} \text{Re}(z_a - \bar{z}_a)[\text{log}(z_a - \bar{z}_a) - 1] > \mathbf{A}^T \mathbf{i}_3. \tag{23}
\]

Case 3: \( \hat{\mathbf{m}}_3 \)

\[
\mathbf{f}(z) = \frac{1}{2\pi i} \text{Re}(z_a - \bar{z}_a) > \mathbf{A}^T \mathbf{i}_2, \tag{24}
\]

where

\[
z_a = x_1 + \mu_a x_2, \quad \hat{\mathbf{p}} = (\hat{\mathbf{f}}_1, \hat{\mathbf{f}}_2, \hat{\mathbf{m}}_2 - \hat{\mathbf{m}}_1)^T, \quad \mathbf{i}_2 = (0100)^T, \quad \mathbf{i}_3 = (0010)^T. \tag{25}
\]

In the above, \( \hat{\mathbf{f}} = (\hat{\mathbf{f}}_1, \hat{\mathbf{f}}_2, \hat{\mathbf{f}}_3) \) and \( \hat{\mathbf{m}} = (\hat{\mathbf{m}}_1, \hat{\mathbf{m}}_2, \hat{\mathbf{m}}_3) \) are the concentrated force and moment applied at point \( \mathbf{x} = (\hat{x}_1, \hat{x}_2) ; \mu_a, \alpha = 1,2,3,4, \) are the material eigenvalues; the superscript \( T \) denotes the transpose of a matrix or a vector; the angular bracket stands for the diagonal matrix whose components vary according to the subscript \( \alpha \).

3.2 Laminates with holes/cracks (Hwu, 2005)

Case 1: \( \hat{\mathbf{f}}_1, \hat{\mathbf{f}}_2, \hat{\mathbf{m}}_1, \hat{\mathbf{m}}_2 \)

\[
\mathbf{f}(\zeta) = \frac{1}{2\pi i} \left\{ \sum_{k=1}^{4} \text{Re}(\zeta_{a} - \bar{\zeta}_a) > \mathbf{B}^{-1} \mathbf{B} \mathbf{I}_k \mathbf{A}^T \hat{\mathbf{p}} \right\}. \tag{26}
\]
Case 2:  $\hat{f}_3$

$$f(\hat{\zeta}) = \langle \zeta_a - \hat{\zeta}_a \rangle \log(\zeta_a - \hat{\zeta}_a) > q_2,$$

$$- \sum_{k=1}^{4} < (\zeta_a - \hat{\zeta}_a - 1 - \hat{\zeta}_a) \log(\zeta_a - \hat{\zeta}_a) > B^T \overline{B} \zeta_k < \zeta_a > q_2,$$

$$+ < (\zeta_a - \hat{\zeta}_a) > q_4^* - \sum_{k=1}^{4} < (\zeta_a - \hat{\zeta}_a) > B^T \overline{B} \zeta_k q_4^*,$$

$$- < (\zeta_a - \hat{\zeta}_a) > q_4^* + \sum_{k=1}^{4} < (\zeta_a - \hat{\zeta}_a) > B^T \overline{B} \zeta_k q_4^*.$$

(27)

Case 3:  $\hat{m}_3$

$$f(\hat{\zeta}) = \frac{1}{\zeta_a - \hat{\zeta}_a} > q_3^*, - \sum_{k=1}^{4} \frac{1}{\zeta_a - \hat{\zeta}_a} > B^T \overline{B} \zeta_k^* q_3^*,$$

(28)

where

$$q_c = \langle c_a > q_2, \quad q^*_c = \langle c_a (\log c_a - 1) > q_2,$$

$$q^{**}_c = \langle c_a (\log (\hat{\zeta}_a) > q_2,$$

$$q^*_3 = \frac{\hat{\zeta}_a}{c_a (\zeta_a - \hat{\zeta}_a)} q_3,$$

$$q_1 = \frac{1}{2\pi} A^T \hat{p}, \quad q_2 = \frac{\hat{f}_1}{2\pi} A^T i_3, \quad q_3 = \frac{\hat{m}_3}{2\pi} A^T l_3.$$

4 CONCLUDING REMARKS

With the boundary integral equations and fundamental solutions provided in this paper, the boundary element for the bending-stretching coupling analysis of composite laminates is expected to be established by following the standard procedure. From the formulation obtained in (19) and (20), we see that the possible trouble for the boundary element formulation may come from the area integration, the corner forces, and the partial derivatives of the displacements. Moreover, since the fundamental solutions provided in (21-29) are written in complex form, follow-up derivation to suit for the boundary integral equations is also necessary.

REFERENCES:


