VIBRATION ANALYSIS OF STIFFENED COMPOSITE MULTICELL WING STRUCTURES

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1. SUMMARY
A dynamic model for the stiffened composite multicell wing structures is developed in this paper. The composite wing structure is modeled incorporating the effects of bending-torsion coupling, warping restraint, transverse shear deformation, shape of airfoil, rotary inertia, etc. To avoid the complexity of formulation, the matrix form representation is introduced, which then makes most of the equations for the vibration analysis bear the same form as those of the classical beam theory. Thus, by following the classical approach, the composite wing structures can be studied analytically.

2. INTRODUCTION
Over the last five decades several different structural models have been proposed to study the composite wing structures such as the classical beam model, the coupled bending-torsion beam model, and the refined model considering the warping restraint and/or transverse shear deformation and/or shell bending strain and/or cross-sectional materials and geometries, etc. To have an improved prediction for the mechanical behavior of the stiffened composite multicell wing structures, the comprehensive refined model incorporating the various elastic and structural couplings, the warping restraint, the transverse shear deformation, the shape of airfoil effects and so on should naturally be a better choice for the structural analysis. However, it is expectable that the more comprehensive the model is, the more difficult to get meaningful and useful analytical results. Hence even the comprehensive model was proposed around ten years ago, most of the analytical works of wing structural analysis still used the classical beam model if the ignored effects will not induce drastic change of the results. In order to benefit from the accuracy of the comprehensive model as well as the simplicity of the classical beam model, in this paper we re-derive the comprehensive model into a simple and elegant form by using the matrix form representation. Due to its simplicity, the vibration analysis by the comprehensive model, which is expected to be complicated, now becomes as simple as that of classical beam model.

3. MATHEMATICAL FORMULATION
Due to the closely spaced stringers and the transverse stiffening members like wing spars and ribs, in aircraft analysis it is usually assumed that the wing chordwise section is rigid [1]. The displacement field consistent with the chordwise-rigid postulation and the basic assumptions for the composite sandwich plates can be written as [2]
\[ u(x, y, z, t) = z \theta(y, t), \]
\[ v(x, y, z, t) = v_0(y, t) + z \{ \beta_x(y, t) + x \beta_y(y, t) \}, \tag{1} \]
\[ w(x, y, z, t) = w_y(y, t) - x \theta(y, t), \]

where \( u, v, w \) are the displacement components in the directions of \( x \) (chordwise), \( y \) (spanwise) and \( z \) (thicknesswise), respectively (see Figure 1). \( t \) denotes the time variable. \( v_0 \) is the mid-plane displacements in \( y \) direction. \( w_y \) denotes the deflection (positive upward) measured at the line of the reference axis; \( \theta \) is the rotation angle with respect to \( x \)-axis due to the twist around the reference axis (positive nose up), i.e., \( \beta_x = \theta \). \( \beta_y \) denotes the rotation angle with respect to \( y \) axis measured at the reference axis and \( \beta_r \) stands for the rate of angle change in the \( x \)-direction. Thus, \( \beta_y = \beta_y + x \beta_r \). By the assumption given in (1) and using the matrix notation, the equations of motion, constitutive relations and boundary conditions can be written in a compact matrix form as follows [3]:

\[
F - F_0 + p = I_0 \delta, \\
F = K_1 \delta + K_2 \delta', \quad F_0 = K_0 \delta + K_1' \delta', \\
F = \tilde{F}, \quad \text{or} \quad \delta = \tilde{\delta}, \quad \text{along} \quad y = \text{constant}
\]

where

\[
\begin{align*}
F & = \begin{pmatrix} \tilde{N}_y \\ \tilde{Q}_y \\ \tilde{M}_y \\ \lambda \tilde{M}_y' \end{pmatrix}, \\
F_0 & = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tilde{M}_y + \tilde{Q}_y' \end{pmatrix}, \\
p & = \begin{pmatrix} \tilde{p}_y \\ \tilde{p}_x \\ \tilde{m}_y \\ \beta_f \end{pmatrix}, \\
\delta & = \begin{pmatrix} \theta \\ \beta_f \end{pmatrix}
\end{align*}
\]

and

\[
K_2 = \begin{pmatrix}
\tilde{A}_{22} & 0 & 0 & 0 \\
0 & \tilde{A}_{44} & -\tilde{A}_{44} & 0 \\
0 & \tilde{D}_{26} + \tilde{A}_{44} & \tilde{D}_{26} & \tilde{D}_{26} \\
\lambda \tilde{B}_{22} & 0 & \lambda \tilde{D}_{26} & \lambda \tilde{D}_{26} \\
\end{pmatrix}, \\
K_1 = \begin{pmatrix}
0 & 0 & 0 & \tilde{B}_{26} \\
0 & 0 & 0 & \tilde{A}_{44} \\
0 & 0 & 0 & \tilde{D}_{26} \\
0 & 0 & 0 & \lambda \tilde{D}_{26} \\
\end{pmatrix}, \\
K_0 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}, \\
I_0 = \begin{pmatrix}
m & 0 & 0 & m \gamma_c \\
0 & m & -m \gamma_c & 0 \\
0 & -m \gamma_c & I_y & 0 \\
m \gamma_c & 0 & 0 & I_{xz} \\
0 & 0 & I_{xz} & I_{xz^2} \end{pmatrix}
\]

Substituting the second and third equations of (2) into the first equation of (2), we can then write down the governing equations for the stiffened composite wing structures in terms of the basic function vector \( \delta' \) as

\[
K_2 \delta'(y, t) + (K_1 - K_0') \delta'(y, t) - K_0 \delta(y, t) + p(y, t) = I_0 \delta(y, t). \tag{5}
\]

If the wing is fixed at the root \((y = 0)\) and free at the tip \((y = L)\), by the last two equations of (2) the boundary conditions of the wing structures can now be written as

\[
\delta(0) = 0, \quad \text{and} \quad K_0 \delta(L) + K_2 \delta'(L) = 0. \tag{6}
\]

To know the natural frequency and its associated vibration mode of the stiffened composite
wing structures, we consider the case that the external forces \( \tilde{p} \) and \( \tilde{p}_y \), torsional moment \( \tilde{p}^T \) as well as the distributed moments \( \tilde{m}_x, \tilde{m}_y, \tilde{m}_z \) are all zero, i.e., the force vector \( \mathbf{p}=0 \) in (3)\(_3\). To find the natural modes of vibration, the usual way is the method of separation of variables. By this method we write the deflection \( \delta(y,t) \) as a product of a function \( \Delta(y) \) of the spatial variables only and a function \( f(t) \) depending on time only. Furthermore, because of free vibration \( f(t) \) is harmonic and of frequency \( \omega \). Thus,

\[
\delta(y,t) = \Delta(y)e^{jwt},
\]

where

\[
\Delta(y) = \begin{bmatrix}
V_0(y) \\
W_f(y) \\
\Theta(y) \\
B_{iy}(y) \\
B_{ry}(y)
\end{bmatrix}.
\]

Through the use of (7), the equation of motion (5) can easily be reduced to a system of ordinary differential equations

\[
K_2\Delta''(y) + (K_1 - K_1^T)\Delta'(y) - K_0\Delta(y) + \omega^2 I_0 \Delta(y) = 0,
\]

which can be solved by letting

\[
\Delta(y) = d e^{\gamma y}.
\]

Substituting (10) into (9), we get

\[
\left[ K_2r^2 + (K_1 - K_1^T)r - K_0 + \omega^2 I_0 \right] d = 0,
\]

whose non-vanishing solutions exist only when the determinant of the coefficient matrix of \( d \) becomes zero. As shown in (4), the coefficient matrix of \( d \) is a 5\times5 matrix for the general cases. Thus, the determinant is a 10th-order polynomial equation which will have 10 roots \( r_i(\omega), i = 1, 2, ..., 10 \). Each of the roots has an associated eigenvector \( d_i(\omega) \) determined from (11). Linear superposition of these ten homogeneous solutions now gives us

\[
\Delta(y) = \sum_{i=1}^{10} k_i d_i e^{\gamma_i y}.
\]

Substituting (12) and (7) into the boundary conditions (6) will then set a system of ten simultaneous linear algebraic equations with ten unknown coefficients \( k_i \) as

\[
\begin{bmatrix}
K_1D < e^{\gamma_1 y} > + K_2D < r_i e^{\gamma_i y} > \\
D
\end{bmatrix} k = 0,
\]

where

\[
D = \begin{bmatrix}
d_1 & d_2 & \ldots & d_{10}
\end{bmatrix}, \quad k = \begin{bmatrix}
k_1 \\
k_2 \\
\vdots \\
k_{10}
\end{bmatrix},\]

and the angular bracket \( <> \) stands for a diagonal matrix in which each component is varied according to its subscript \( i \), e.g., \( < r_i e^{\gamma_i y} >= \text{diag}[r_1 e^{\gamma_1 y}, r_2 e^{\gamma_2 y}, \ldots, r_{10} e^{\gamma_{10} y}] \).

Because both of the eigenvalues \( r_i \) and the eigenvectors \( d_i \) are functions of the natural frequency \( \omega \), the coefficient matrix of \( k \) in (13) is a function of the natural frequency \( \omega \). Again, non-vanishing solutions exist only when the determinant of the coefficient matrix of \( k \) becomes zero, by which we can then obtain the natural frequencies of the stiffened composite wing structures. With the determined natural frequency \( \omega \), the
coefficients $k_i$ can be calculated from (13) as the eigenvector, and hence the natural vibration mode shapes of the composite wing structures are obtained from (12).

It has also been proved [3] that the family of natural vibration mode shapes $\Delta_j(y)$ can constitute a complete set of orthonormal modes, i.e.,

$$\int_0^L \Delta_j^T(y) I_0 \Delta_i(y) dy = \delta_{ij},$$

(15)

where $\delta_{ij}$ is the Kronecker delta. Unlike the usual orthogonality conditions that only the lateral deflection is considered, the orthogonality found in (15) shows that the complete set includes not only the mode shapes of the deflection but also the mode shapes of all the other basic functions.

After finding the orthogonality relation (15), the expansion theorem may be used to obtain the system response by modal analysis. Using the expansion theorem we write the solution of (5) as a superposition of the natural modes $\Delta_j(y)$ multiplying corresponding time-dependent generalized coordinates $\eta_j(t)$, hence,

$$\vec{\delta}(y,t) = \sum_{j=1}^{\infty} \Delta_j(y) \eta_j(t).$$

(16)

Introducing (16) into the governing equation (5) and using the orthogonality relation (15), an infinite set of uncoupled second order ordinary differential equation system can be obtained as

$$\ddot{\eta}_j(t) + \omega_j^2 \eta_j(t) = N_j(t) \quad j = 1, 2, \ldots,$$

(17)

where $N_j(t)$ denotes a generalized force associated with the generalized coordinate $\eta_j(t)$ and is related to the load vector $p$ by

$$N_j(t) = \int_0^L \Delta_j^T(y)p(y,t) dy.$$ 

(18)

3. NUMERICAL EXAMPLES

Comparison with the existing solutions has been done for several special cases such as a bending-torsion coupled cantilevered composite beam and a cantilever composite sandwich beam, etc., and the results show that our solutions of the natural frequencies agree well with the existing experimental and analytical solutions [3]. After verifying our results by using the flat composite beams, a composite wing structure with NACA 2412 airfoil is also analysed and compared with the results obtained from the commercial finite element software ANSYS and their results are also well agree with each other [3]. To save the space of this paper, here we just list one comparison result for the natural frequencies (see Table 1) and one new result for the case of vibration suppression (see Figure 2). Detailed numerical data and discussions for these results can be found in [3] and [4].

4. CONCLUSIONS

By using the matrix form representation, the vibration analysis shown in this paper becomes simple and elegant. Many equations if written in scalar form may be very complicated and not easy to be solved analytically. With matrix form representation, most of the equations bear the same form as those of the classical beam theory. Although the matrix form looks simple, the comprehensive model presented in this paper is quite general and can cover several different simplified models considered in the literature.

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5. REFERENCES


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a: The natural frequencies by FEM from 4th mode to 7th mode are, respectively, 1325.8, 1468.1, 1983.9, 2029.8.
Figure 1. Geometry of the stiffened composite multicell wing.

Figure 2. Tip displacement response of the first mode of the composite wing with noise (example of vibration suppression)