A NOVEL FORMALISM FOR THE CLASSICAL LAMINATION THEORY

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ABSTRACT: Based upon the knowledge of Stroh formalism and Lekhnitskii formalism for two-dimensional anisotropic elasticity as well as the complex variable formalism developed by Lekhnitskii for plate bending problems, in this paper a novel formalism for the classical lamination theory is established. The key feature that makes Stroh formalism more attractive than Lekhnitskii formalism is that the former possesses the eigen-relation which relates the eigen-modes of stress functions and displacements to the material properties. To retain this special feature, the associated eigen-relation has also been obtained for the present formalism. By intentional rearrangement, this new formalism and its associated relations look almost the same as those for the two-dimensional problems. Therefore, almost all the techniques developed for the two-dimensional problems can now be applied to the plate bending problems of composite laminates. Thus, many unsolved plate bending problems can now be solved if their corresponding two-dimensional problems have been solved successfully.

1. INTRODUCTION

Although the classical lamination theory was developed long time ago, it is not easy to apply this theory to find a solution for the problem with curvilinear boundaries. For example, the problems with holes/cracks/inclusions, which have been discussed and solved vastly in two-dimensional problems, are still difficult problems for the plates under out-of-plane bending actions. Because this kind of problems was solved by the complex variable formulation in two-dimensional deformation, it is hoped that similar formulation can be developed for the classical lamination theory. Tracing the literature, we found that Lekhnitskii has ever developed a complex variable formalism for the plate bending problems (Lekhnitskii, 1938), and used his formalism to solve the problems of orthotropic plates containing circular holes (Lekhnitskii, 1968). After that, very few contributions can be found in the literature for the improvement of complex variable formulation.

Unlike the plate bending problems, the complex variable methods in two-dimensional anisotropic elasticity have reached a big step by connecting Lekhnitskii formulation and Stroh formalism (Suo, 1990; Hwu, 1996; Barnett and Kirchner, 1997; Ting, 1999; Yin, 2000a, 2000b), especially when the book (Ting, 1996) concerning Stroh formalism was published. In this paper, by carefully reviewing Lekhnitskii formalism for both the two-dimensional and
plate bending problems and catching the spirit of Stroh formalism for two-dimensional problems, we develop a novel formalism for the classical lamination theory. Both of the symmetric and unsymmetric laminates are considered in our formalism. Due to the relative simplicity of symmetric laminates, the novel formalism for symmetric laminates look very like those of Stroh formalism for two-dimensional linear anisotropic elasticity. Hence, almost all the mathematical techniques developed for two-dimensional problems can lend to the problems of symmetric laminates. On the other hand, a novel formalism for the unsymmetric laminates has also been developed in this paper by using a different approach, which was introduced by Lu and Mahrenholtz (1994) and corrected by (Cheng and Reddy, 2001). Because these two formalisms are developed by different approaches, their equivalence is important for the future development just like the connection between Lekhnitskii and Stroh formalisms for two-dimensional problems.

2. CLASSICAL LAMINATION THEORY

To describe the overall properties and macromechanical behavior of a laminate, the most popular way is the classical lamination theory (Jones, 1974). According to the observation of actual mechanical behavior of laminates, following assumptions are made in this theory. (a) The laminate consists of perfectly bonded laminae and the bonds are infinitesimally thin as well as non-shear-deformable. Thus, the displacements are continuous across lamina boundaries so that no lamina can slip relative to another. (b) A line originally straight and perpendicular to the middle surface of the laminate remains straight and perpendicular to the middle surface of the laminate when the laminate is deformed. In other words, the transverse shear strains are ignored, i.e., \( \gamma_{xz} = \gamma_{yz} = 0 \). (c) The normals have constant length so that the strain perpendicular to the middle surface is ignored, i.e. \( \varepsilon_z = 0 \).

Based upon the above assumptions, the laminate displacements \( u, v, \) and \( w \) in the \( x, y \) and \( z \) directions can be expressed as

\[
\begin{align*}
  u(x, y, z) &= u_0(x, y) - z \frac{\partial w(x, y)}{\partial x}, \\
  v(x, y, z) &= v_0(x, y) - z \frac{\partial w(x, y)}{\partial y}, \\
  w(x, y, z) &= w_0(x, y),
\end{align*}
\]

where \( u_0, v_0 \) and \( w_0 \) are the middle surface displacements. Based upon the displacement fields assumed in (2.1) and the small deformation assumption, the laminate strains can then be written in terms of the mid-surface strains \( (\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0) \) and plate curvature \( (\kappa_x, \kappa_y, \kappa_{xy}) \) as follows

\[
\begin{align*}
  \varepsilon_x &= \varepsilon_x^0 + z\kappa_x, \quad \varepsilon_y = \varepsilon_y^0 + z\kappa_y, \quad \gamma_{xy} = \gamma_{xy}^0 + z\kappa_{xy},
\end{align*}
\]

where

\[
\begin{align*}
  \varepsilon_x^0 &= \frac{\partial u_0}{\partial x}, \quad \varepsilon_y^0 = \frac{\partial v_0}{\partial y}, \quad \gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}, \\
  \kappa_x &= -\frac{\partial^2 w}{\partial x^2}, \quad \kappa_y = -\frac{\partial^2 w}{\partial y^2}, \quad \kappa_{xy} = -2\frac{\partial^2 w}{\partial x \partial y}.
\end{align*}
\]

Like the classical plate theory, the thickness of the laminate is considered to be small compared to its other dimensions. Therefore, instead of dealing the stress distribution across the laminate thickness, an integral equivalent system of forces and moments acting on the
laminate cross section is used in the classical lamination theory. Hence, the constitutive laws relating the stresses and strains in an elastic body are now expressed in terms of the resultant forces/moments and mid-surface strains/curvatures. For a composite laminate, the constitutive laws are usually expressed as follows.

\[
\begin{pmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_y \\
M_{xy}
\end{pmatrix} = 
\begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\
B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66}
\end{bmatrix} 
\begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0 \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix},
\]

(2.4)

where \(N_x, N_y, N_{xy}\) are the resultant forces and \(M_x, M_y, M_{xy}\) are the resultant moments. \(A_{ij}, B_{ij}\) and \(D_{ij}\) are, respectively, the extensional, coupling and bending stiffness matrices, and are determined by

\[
A_{ij} = \sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_k - h_{k-1}),
\]

\[
B_{ij} = \frac{1}{2} \sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_k^2 - h_{k-1}^2),
\]

\[
D_{ij} = \frac{1}{3} \sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_k^3 - h_{k-1}^3),
\]

(2.5)

where \(h_k\) and \(h_{k-1}\) denotes, respectively, the location of the bottom and top surface of the \(k\)th lamina (Figure 1). \((\overline{Q}_{ij})_k\) is the transformed stiffness matrix of the \(k\)th lamina.

Figure 1. Laminate geometry, resultant forces and moments
The equilibrium equations concerning the balance of forces acting upon the structures, which should be independent of the material types, can be described by using the resultant forces and moments as

\[
\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \quad \frac{\partial N_y}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0,
\]

\[
\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0,
\]

where \( q \) is the lateral distributed load applied on the laminates. Note that the third equation of (2.6) represents the forces equilibrium in the thickness direction, which is usually written in terms of the transverse forces \( Q_x \) and \( Q_y \) as

\[
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0.
\]

The moment equilibrium in the \( x \)- and \( y \)-directions shows that the transverse shear forces are related to the bending moments by

\[
Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}, \quad Q_y = \frac{\partial M_y}{\partial x} + \frac{\partial M_{yx}}{\partial y}.
\]

Substituting (2.8) into (2.7), we may get the third equation of (2.6).

**Governing Equations:**

The basic equations for the laminated plates are given in (2.1) for displacement fields, (2.2) and (2.3) for the strain-displacement relations, (2.4) for the constitutive laws, and (2.6) for the equilibrium equations. Among these basic equations, only the constitutive laws depend on the material properties. All the other equations are exactly the same as those of the classical plate theory. To get governing equations satisfying all these basic equations, we first use (2.3) to express the midplane strains \( \varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0 \) and curvatures \( \kappa_x, \kappa_y, \kappa_{xy} \) in terms of the midplane displacements \( u_0, v_0 \) and \( w \), then use (2.4) to express the resultant forces \( N_x, N_y, N_{xy} \) and moments \( M_x, M_y, M_{xy} \) in terms of the midplane displacements. After these direct substitutions, the three equilibrium equations (2.6) can now be written in terms of three unknown displacement functions \( u_0, v_0 \) and \( w \) as

\[
A_{11} \frac{\partial^2 u_0}{\partial x^2} + 2A_{16} \frac{\partial^2 u_0}{\partial x \partial y} + A_{66} \frac{\partial^2 u_0}{\partial y^2} + A_{16} \frac{\partial^2 v_0}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 v_0}{\partial x \partial y} + A_{26} \frac{\partial^2 v_0}{\partial y^2} = 0,
\]

\[
- B_{11} \frac{\partial^3 w}{\partial x^3} - 3B_{16} \frac{\partial^3 w}{\partial x^2 \partial y} - (B_{12} + 2B_{66}) \frac{\partial^3 w}{\partial x \partial y^2} - B_{26} \frac{\partial^3 w}{\partial y^3} = 0,
\]

\[
A_{16} \frac{\partial^2 u_0}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 u_0}{\partial x \partial y} + A_{26} \frac{\partial^2 u_0}{\partial y^2} + A_{66} \frac{\partial^2 v_0}{\partial x^2} + 2A_{26} \frac{\partial^2 v_0}{\partial x \partial y} + A_{22} \frac{\partial^2 v_0}{\partial y^2} = 0,
\]

\[
- B_{16} \frac{\partial^3 w}{\partial x^3} - (B_{12} + 2B_{66}) \frac{\partial^3 w}{\partial x^2 \partial y} - 3B_{26} \frac{\partial^3 w}{\partial x \partial y^2} - B_{22} \frac{\partial^3 w}{\partial y^3} = 0,
\]
which are the governing equations for the laminated plates.

The governing equations shown in (2.9a,b,c) are system of partial differential equations with three unknown functions $u_0, v_0$ and $w$. Due to the mathematical complexity of these equations, it is not easy to get solutions by solving these partial differential equations. In practical engineering applications, it is common to have a symmetric laminate or to construct a balanced laminate. In those cases the coupling stiffness components like $B_{ij}$ and/or $A_{16}, A_{26}$ and/or $D_{16}, D_{26}$ will be zero, and equations (2.9a,b,c) will be drastically simplified.

### Boundary Conditions:

For the general cases of laminated plates, the in-plane and bending problems will couple each other. Hence, every boundary of the plates should be described by four prescribed values. Two of them correspond to the in-plane problems and the other two correspond to the bending problems. Generally, they may be expressed as

$$
u_n = u_n, \quad \nu_n = v_n, \quad \nu_n = k_n u_n, \quad \nu_n = k_n v_n, \quad (2.10)$$

where $V_n$ is the well known Kirchhoff force of classical plate theory, or called effective shear force defined by

$$V_n = Q_n + \frac{\partial M_n}{\partial t}, \quad (2.11)$$

The subscripts $n$ and $t$ denote, respectively, the directions normal and tangent to the boundary. The overhat denotes the prescribed value. The values in the n-t coordinate can be calculated from the values in the x-y coordinate according to the transformation laws.

### 3. A NOVEL FORMALISM FOR SYMMETRIC LAMINATES

For symmetric laminates, the coupling stiffness matrix $B_{ij}$ defined in (2.5.2) will be identical to zero. With this arrangement, the governing equation shown in (2.9) can then be decoupled into the in-plane problem and plate bending problem. If we consider only the plate bending problem, the governing equation can be expressed in terms of the mid-plane lateral deflection $w$ as

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4} = q. \quad (3.1)$$

The deflection $w$ can be determined by solving this partial differential equation with the satisfaction of the boundary conditions set for the considered problems. After finding the
deflection, all the other physical values such as the in-plane displacements \((u, v)\), bending moments \((M_x, M_y, M_{xy})\), transverse shear forces \((Q_x, Q_y)\), and internal stresses \((\sigma_x, \sigma_y, \tau_{xy})\) can all be obtained through the use of their relations with the deflection.

Although it seems not difficult to get a general solution for the deflection to satisfy the governing equation (3.1), it is really not easy to find a unique solution satisfying the boundary conditions for the complicated geometrical boundaries by using the conventional methods introduced in most of the textbooks of plates such as (Szilard, 1985). In order to solve the problems with complicated geometrical boundaries, recently I developed a Stroh-like complex variable formalism for the bending theory of anisotropic plates (Hwu, 2001), which can be directly applied to the cases of symmetric laminates. In this formalism, the general solutions satisfying the governing equation (1) can be expressed as follows

\[
\left. \begin{array}{l}
{w} = w_0 + 2 \text{Re} \{ c_1 w_1(z_1) + c_2 w_2(z_2) \}, \\
\end{array} \right. 
\]

where \( Re \) stands for the real part of a complex number; \( w_0 \) is a particular solution of eqn.(1), whose form depends on the load distribution \( q \) on a plate surface; \( w_1(z_1) \) and \( w_2(z_2) \) are arbitrary analytic functions of complex variables

\[
z_k = x + \mu_k y, \quad k = 1,2, 
\]

where \( \mu_k, k=1,2 \), are the material eigenvalues which have been assumed to be distinct in the general expression (3.2). The determination of the material eigenvalues and their associated eigenvectors will be described later. The coefficients \( c_1 \) and \( c_2 \) will also be defined after introducing the material eigenvectors.

By introducing the stress function vector \( \phi \) and the slope vector \( \alpha \) as

\[
\phi = \left\{ \phi_1, \phi_2 \right\} = \left\{ -\int M_x \, dx, -\int M_y \, dy \right\}, \quad \alpha = \left\{ \alpha_1, \alpha_2 \right\} = \left\{ -\frac{\partial w}{\partial y}, -\frac{\partial w}{\partial x} \right\}, 
\]

and using the general solution (3.2), a Stroh-like formalism can be written as (Hwu, 2001)

\[
\phi = \phi_0 + 2 \text{Re} \{ A_s w'(z) \}, \quad \alpha = \alpha_0 + 2 \text{Re} \{ B_s w'(z) \},
\]

where \( \phi_0 \) and \( \alpha_0 \) are the particular solutions related to the lateral load distribution \( q \); \( w'(z) \) is a function vector defined as

\[
w'(z) = \left\{ w_1'(z_1), w_2'(z_2) \right\},
\]

in which the prime ('') denotes differentiation with respect to its argument \( z_k \). \( \mu_k \), and \( A_s, B_s \) are the material eigenvalues and eigenvector matrices, which can be determined from the following eigen-relation:

\[
N_s \xi = \mu \xi, 
\]

where

\[
N_s = \begin{bmatrix}
N_1 & N_2 \\
N_3 & N_1^T
\end{bmatrix}, \quad \xi = \begin{bmatrix}
\mathbf{a} \\
\mathbf{b}
\end{bmatrix},
\]

and

\[
N_1 = -T^{-1} R^T, \quad N_2 = T^{-1} = N_2^T, \quad N_3 = RT^{-1} R^T - Q = N_3^T.
\]

In the above eigen-relation, the three 2×2 real matrices \( Q, R \) and \( T \) are defined as
\[
\begin{bmatrix}
D^*_2 & -\frac{1}{2} D^*_6 \\
-\frac{1}{2} D^*_2 & D^*_6
\end{bmatrix}, \quad 
\begin{bmatrix}
-\frac{1}{2} D^*_6 & D^*_12 \\
\frac{1}{4} D^*_6 & -\frac{1}{2} D^*_16
\end{bmatrix}, \quad 
\begin{bmatrix}
\frac{1}{4} D^*_66 & \frac{1}{2} D^*_16 \\
-\frac{1}{2} D^*_16 & D^*_14
\end{bmatrix}, \quad (3.8)
\]

in which \(D^*_y\) is the components of the inverse bending stiffness matrix \(D_y\), i.e.,

\[
\begin{bmatrix}
D^*_11 & D^*_12 & D^*_16 \\
D^*_12 & D^*_22 & D^*_26 \\
D^*_16 & D^*_26 & D^*_66
\end{bmatrix} = D^{-1}.
\quad (3.9)
\]

It has been proved that the material eigenvalues \(\mu_k\) cannot be real due to the positive definiteness of strain energy and they will appear in two pairs of complex conjugates (Lekhnitskii, 1968). If the eigenvalues are assumed to be distinct and are arranged in the order that \(\mu_1\) and \(\mu_2\) are those with positive imaginary parts, their associated eigenvectors will be independent each other and the eigenvector matrices \(A_s\) and \(B_s\) are defined as

\[
A_s = [a_1, a_2], \quad B_s = [b_1, b_2]. \quad (3.10)
\]

For the convenience of readers’ reference, the explicit expressions of the fundamental matrix \(N_s\) and the eigenvectors \(a_k\) and \(b_k\) obtained in (Hwu, 2001) are listed below.

\[
N_1 = \begin{bmatrix}
-2D_{26} & 1 \\
D_{12} & D_{22}
\end{bmatrix}, \quad N_2 = \begin{bmatrix}
-2D_{16} + \frac{2D_{12}D_{26}}{D_{22}} & 4D_{26} - \frac{4D_{26}^2}{D_{22}} \\
-2D_{16} & D_{11} - \frac{D_{12}^2}{D_{22}}
\end{bmatrix}, \quad N_3 = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix},
\]

and

\[
a_k = c_k \begin{bmatrix} h_k \\ g_k \end{bmatrix}, \quad b_k = c_k \begin{bmatrix} -\mu_k \\ 1 \end{bmatrix}, \quad (3.12a)
\]

where

\[
h_k = D_{12} + D_{22}\mu_k^2 + 2D_{26}\mu_k, \quad g_k = \frac{D_{11}}{\mu_k} + D_{12}\mu_k + 2D_{16}, \quad (3.12b)
\]

\[
c_k^2 = \frac{1}{2(g_k - \mu_k h_k)}, \quad k = 1, 2. \quad (3.12c)
\]

The key feature of this novel formalism is that it looks almost the same as Stroh formalism for two-dimensional linear anisotropic elasticity. Hence, almost all the mathematical techniques developed for two-dimensional problems can lend to the plate bending problems. By simple analogy, some of the practically important problems whose explicit analytical solutions have not been found in the literature, like the problems with elliptical holes or cracks or inclusions, can now be solved exactly by using this novel formalism (Hsieh and Hwu, 2001).

**Moments and Transverse Shear Forces**

In practical applications, one is usually interested in the moments and shear forces induced in the forced laminates. To find the moments and transverse shear forces, the conventional way is calculating the moments \((M_x, M_y, M_{xy})\) and transverse shear forces \((Q_x, Q_y)\) or
effective transverse shear forces \((V_x, V_y)\) by using their relations with the deflection \(w\), then using the transformation law to find their values in the principal directions, i.e. \(M_n, M_t, M_m, Q_n, Q_t, V_n, V_t\). Since in our newly developed Stroh-like formalism, the final results are expressed in terms of the stress function vector \(\phi\) and slope vector \(\alpha\), it is hoped that the moments and the transverse shear forces can be found directly from \(\phi\). To achieve this goal, use of the equilibrium equations and the concept of surface traction in two-dimensional problems, we obtain (Hsieh and Hwu, 2001)

\[
M_n = -s^T \phi_{,n}, \quad M_t = -n^T \phi_{,n}, \quad M_m = \frac{1}{2} (s^T \phi_{,n} + n^T \phi_{,n})
\]

\[
Q_n = -\frac{1}{2} (s^T \phi_{,n} - n^T \phi_{,n}), \quad Q_t = \frac{1}{2} (s^T \phi_{,nn} - n^T \phi_{,nn}).
\]  

(3.13a)

\[
V_n = n^T \phi_{,nx}, \quad V_t = s^T \phi_{,nn},
\]

where

\[
s^T = [-\sin \theta \quad \cos \theta], \quad n^T = [\cos \theta \quad \sin \theta],
\]  

(3.13b)

and \(\theta\) denotes the angle between the normal \(n\) and \(x\)-axis (see Figure 1).

4. A NOVEL FORMALISM FOR UNSYMMETRIC LAMINATES

Through the connection between Stroh formalism and Lekhnitskii formalism for the two-dimensional problems, a Stroh-like formalism has been developed in the last section for the plate bending problems of symmetric laminates. However, by comparing the governing equations for the unsymmetric laminates (2.9) and symmetric laminates (3.1), we see that due to the coupling of the in-plane and plate bending problems for unsymmetric laminates, it is really difficult to apply the same approach to unsymmetric laminates. Instead of using the advantages of Stroh-Lekhnitskii’s connection, a direct derivation has been tried recently by Lu and Mahrenholtz (1994) and by Cheng and Reddy (2001). However, due to the complexity, the resemblance is not perfect enough to employ most of the key features of two-dimensional problems. To improve their formalism, similar approach will be used in the following derivation.

For the convenience of later derivation, we will firstly rewrite all the basic equations (2.1)-(2.8) in terms of tensor notation as follows

\[
U_i = u_i + z \beta_i, \quad \beta_1 = -w_{,1}, \quad \beta_2 = -w_{,2},
\]

\[
\varepsilon_{ij} = \epsilon_{ij} + z \kappa_{ij}, \quad \kappa_{ij} = \frac{1}{2} (U_{i,j} + U_{j,i}), \quad \epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}),
\]

\[
N_{ij} = A_{ijkl} \epsilon_{kl} + B_{ijkl} \kappa_{kl}, \quad M_{ij} = B_{ijkl} \epsilon_{kl} + D_{ijkl} \kappa_{kl},
\]

\[
M_{ij,k,l} = 0, \quad M_{ij} + q = 0, \quad Q_{ij} = M_{ij,k,l}, \quad i,j,k,l = 1,2.
\]

(4.1)

Note that in the above tensor notation, we have employed the following conventional replacements,

\[
x \leftrightarrow 1, \quad y \leftrightarrow 2, \quad 11 \leftrightarrow 1, \quad 22 \leftrightarrow 2, \quad 12 \leftrightarrow 6.
\]

Substituting the strains/curvatures and displacements/slopes relations into the constitutive laws, i.e., substituting (4.1)\(_2\) into (4.1)\(_3\), the resultant forces and moments may be expressed in terms of mid-surface displacements \(u_i\) and slopes \(\beta_i\) as

\[
N_{ij} = A_{ijkl} u_{k,l} + B_{ijkl} \beta_{k,l}, \quad M_{ij} = B_{ijkl} u_{k,l} + D_{ijkl} \beta_{k,l}.
\]  

(4.2)

By employing the results of (4.2) to the equilibrium equations (4.1)\(_4\), the governing
equations may also be expressed in terms of the mid-surface displacements \( u_i \) and slopes \( \beta_i \) as
\[
A_{ijkl}u_{klij} + B_{ijkl}\beta_{klij} = 0, \quad B_{ijkl}u_{klij} + D_{ijkl}\beta_{klij} + q = 0. \tag{4.3}
\]
Consider the homogeneous case that no lateral load is applied on the laminates, i.e., \( q=0 \). Because the mid-surface displacements \( u_i \) and slopes \( \beta_i \) depend only on two variables, \( x_1 \) and \( x_2 \), and (4.3) are homogeneous partial differential equations, we may assume that
\[
u_k = a^N_k f(z), \quad \beta_k = a^M_k f(z), \quad z = x_1 + \mu x_2. \tag{4.4}
\]
Substituting (4.4) into (4.3) with \( q=0 \), we obtain
\[
\{Q_A + \mu(R_A + R_A^T)\}a_N + \{Q_B + \mu(R_B + R_B^T)\}a_M = 0, \tag{4.5}
\]
where
\[
Q_A = A_{ikl}, \quad Q_B = B_{ikl}, \quad Q_D = D_{ikl},
\]
\[
R_A = A_{ikl}, \quad R_B = B_{ikl}, \quad R_D = D_{ikl},
\]
\[
T_A = A_{ikl}, \quad T_B = B_{ikl}, \quad T_D = D_{ikl},
\]
\[
a_N = \begin{bmatrix} a^N_1 \\ a^N_2 \end{bmatrix}, \quad a_M = \begin{bmatrix} a^M_1 \\ a^M_2 \end{bmatrix}, \quad \mu^* = \begin{bmatrix} 1 \\ \mu \end{bmatrix}. \tag{4.6}
\]
From the second and third equations of (4.1) and the assumption of the slope \( \beta_k \) given in (4.4), we get
\[
a^M_2 = \mu a^N_1. \tag{4.7}
\]
Equations (4.5) and (4.7) constitute four equations with four unknowns \( a^N_i, a^M_i, a^N_2, a^M_2 \). Thus, the problem is solved in principle. Substituting (4.4) into (4.2), we have
\[
N_{i1} = -\mu b_N f'(z), \quad N_{i2} = b_N f'(z),
\]
\[
M_{i1} = -\mu d^* f'(z), \quad M_{i2} = d f'(z), \tag{4.8}
\]
where
\[
b_N = (R_A + \mu T_A)a_N + (R_B + \mu T_B)a_M
\]
\[
= -\frac{1}{\mu} \{ (Q_A + \mu R_A)a_N + (Q_B + \mu R_B)a_M \}, \tag{4.9}
\]
\[
d = (T_B + \mu R_B)a_N + (T_D + \mu R_D)a_M,
\]
\[
d^* = -\frac{1}{\mu} \{ (Q_B + \mu R_B)a_N + (Q_D + \mu R_D)a_M \}.
\]
Note that the second equality of (4.9) comes from (4.5). Using the relation for the bending moments and transverse shear forces given in the third equation of (4.1), and the definition for the effective transverse shear force given in (2.11), and the results for the bending moments given in (4.8), we obtain
\[
Q_i = \mu (d - d^*) f''(z), \tag{4.10}
\]
\[
V_1 = \mu a_1^T (2d - d^*) f''(z), \quad V_2 = \mu a_2^T (d - 2d^*) f''(z),
\]
where
\[
i_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad i_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{4.11}
\]
With the definitions of \( d \) and \( d^* \) given in (4.9), the second equation of (4.5) leads to
\[
\mu^* (d - d^*) = 0. \tag{4.12}
\]
Substituting (4.8) into the symmetry condition of the twist moments, i.e., $M_{12} = M_{21}$, we have

$$d_1 = -\mu d_2^*.$$  

(4.13)

Combining (4.12) and (4.13), we may express $d^*$ in terms of $d$, or vice versa. Through their relation, we now introduce a new vector $b_M$ as

$$b_M = d + b_0 i_1 = d^* + \frac{b_0}{\mu} i_2,$$  

(4.14)

where

$$b_0 = \frac{1}{2} \mu^{\ast} b_M = \frac{1}{2} (b_1^M + \mu b_2^M) = d_1 + \mu d_2.$$  

(4.15)

By the relation given in (4.14), the expressions for the bending moments and transverse shear forces obtained in (4.8) and (4.10) can now be written as

$$M_{ii} = (-\mu b_M + b_0 i_2) f' (z), \quad M_{12} = (b_M - b_0 i_1) f' (z),$$

$$Q_i = b_0 \left\{ \begin{array}{l} -\mu \\ 1 \end{array} \right\} f'' (z), \quad V_i = \left\{ \begin{array}{l} -\mu^2 b_2^M \\ b_1^M \end{array} \right\} f'' (z).$$  

(4.16)

Observing the results obtained in (4.8) and (4.16), we introduce two stress functions

$$\phi_i = b_i^N f (z), \quad \psi_i = b_i^M f (z).$$  

(4.17)

With the use of these two stress functions, the moments, transverse shear forces and effective transverse shear forces can be expressed as

$$N_{ii} = -\phi_{i,2}, \quad N_{12} = \phi_{1,1},$$

$$M_{ii} = -\psi_{i,2} - \frac{1}{2} \lambda_{ij} \psi_{k,k}, \quad M_{12} = \psi_{1,1} - \frac{1}{2} \lambda_{i2} \psi_{k,k},$$

$$Q_{i1} = -\frac{1}{2} \psi_{k,2}, \quad Q_{22} = \frac{1}{2} \psi_{k,1},$$

$$V_{11} = -\psi_{2,2}, \quad V_{22} = \psi_{1,1},$$

(4.18)

where $\lambda_{ij}$ is the permutation tensor defined as

$$\lambda_{11} = \lambda_{22} = 0, \quad \lambda_{12} = -\lambda_{21} = 1.$$  

(4.19)

Up to now, the formalism is almost complete because the displacements, slopes, moments, and transverse shear forces have all been expressed elegantly in (4.4), (4.17) and (4.18). The eigenvalues $\mu$ and the displacement eigenvectors $a_A$, $a_M$, can be obtained from (4.5) and (4.7), and the stress function eigenvectors $b_A$, $b_M$, can be obtained from (4.9) and (4.14).

From (4.5), the determination of the eigenvalues $\mu$ will lead to an equation of 8th order polynomial, which can be proved to have four pairs of complex conjugates. By arranging the complex eigenvalues whose imaginary parts are positive to be the first four eigenvalues, and superimposing all their corresponding solutions, the solutions shown in (4.4) and (4.17) may now be written in a compact matrix form as

$$\tilde{u} = 2 \text{Re} \{A_u f (z)\}, \quad \tilde{\phi} = 2 \text{Re} \{B_u f (z)\},$$

(4.20)

where

$$\tilde{u} = \left\{ \begin{array}{l} u \\ \beta \end{array} \right\}, \quad \tilde{\phi} = \left\{ \begin{array}{l} \phi \\ \psi \end{array} \right\},$$

(4.21a)
\[ A_u = [a_1 \ a_2 \ a_3 \ a_4], \quad B_u = [b_1 \ b_2 \ b_3 \ b_4], \]
\[ f(z) = \begin{bmatrix} f_1(z_1) \\
 f_2(z_2) \\
 f_3(z_3) \\
 f_4(z_4) \end{bmatrix}, \quad z_k = x_1 + \mu_k x_2 \]
\[ (4.21b) \]

and
\[ u = \{u_1\}^{u_2}, \quad \beta = \{\beta_1\}^{\beta_2}, \quad \phi = \{\phi_1\}^{\phi_2}, \quad \psi = \{\psi_1\}^{\psi_2}, \quad a_k = \{a_{N}\}^{a_{M}}, \quad b_k = \{b_{N}\}^{b_{M}}. \]  
\[ (4.21c) \]

In order to establish an eigenvalue relation like the Stroh formalism for two-dimensional problems, we re-cast \((4.9)_{1}\) and \((4.14)\) with the assist of \((3.3)_{2}\) into
\[ \begin{bmatrix} \mathbf{Q} & -\frac{1}{2} \mathbf{I}_{43} \\
 \mathbf{R}^T & -\mathbf{I} + \frac{1}{2} \mathbf{I}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\
 \mathbf{b} \end{bmatrix} = \mu \begin{bmatrix} -\mathbf{R} & -\mathbf{I} + \frac{1}{2} \mathbf{I}_{44} \\
 -\mathbf{T} & -\frac{1}{2} \mathbf{I}_{34} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\
 \mathbf{b} \end{bmatrix}, \]
\[ (4.22) \]

where
\[ \mathbf{Q} = \begin{bmatrix} Q_A & Q_B \\
 Q_B & Q_D \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} R_A & R_B \\
 R_B & R_D \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} T_A & T_B \\
 T_B & T_D \end{bmatrix}, \]
\[ a = \{a_N\}^{a_{M}}, \quad b = \{b_N\}^{b_{M}}. \]
\[ (4.23) \]

\(\mathbf{I}\) denotes the identity matrix, and \(\mathbf{I}_{mn}\) stands for a matrix with all zero components except the \(mn\) component, for example,
\[ \mathbf{I}_{34} = \begin{bmatrix} 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{I}_{44} = \begin{bmatrix} 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \end{bmatrix}. \]
\[ (4.25) \]

In order to see more clearly the eigen-relation shown in \((4.22)\), we now write down the expressions of \(\mathbf{Q}_A, \mathbf{Q}_B, \ldots, \mathbf{T}_D\) defined in \((4.6)\) as
\[ \mathbf{Q}_A = \begin{bmatrix} A_{11} & A_{16} \\
 A_{16} & A_{66} \end{bmatrix}, \quad \mathbf{Q}_B = \begin{bmatrix} B_{11} & B_{16} \\
 B_{16} & B_{66} \end{bmatrix}, \quad \mathbf{Q}_D = \begin{bmatrix} D_{11} & D_{16} \\
 D_{16} & D_{66} \end{bmatrix}, \]
\[ \mathbf{R}_A = \begin{bmatrix} A_{16} & A_{12} \\
 A_{66} & A_{26} \end{bmatrix}, \quad \mathbf{R}_B = \begin{bmatrix} B_{16} & B_{12} \\
 B_{66} & B_{26} \end{bmatrix}, \quad \mathbf{R}_D = \begin{bmatrix} D_{16} & D_{12} \\
 D_{66} & D_{26} \end{bmatrix}, \]
\[ \mathbf{T}_A = \begin{bmatrix} A_{26} & A_{22} \\
 A_{26} & A_{22} \end{bmatrix}, \quad \mathbf{T}_B = \begin{bmatrix} B_{26} & B_{22} \\
 B_{26} & B_{22} \end{bmatrix}, \quad \mathbf{T}_D = \begin{bmatrix} D_{26} & D_{22} \\
 D_{26} & D_{22} \end{bmatrix}. \]
\[ (4.26) \]

By expanding \((4.22)\) with the assist of \((4.26)\), we observe that the 2nd and 5th equations of \((4.22)\) will ensure the equality \(b_1^N = -\mu b_2^N\), which is also the consequence of the symmetry of in-plane forces, i.e., \(N_{12} = N_{21}\) by \((4.8)_{1}\). Moreover, it is observed that the 4th and 7th equations of \((4.22)\) are identical, which has also been noticed by Cheng and Reddy (2001). Due to the equivalence of the 4th and 7th equations, only seven independent equations remain in \((4.22)\). The extra independent equation may come from the equality of \((4.7)\), which is a result of thin plate Kirchhoff assumption because the slopes \(\beta_x\) and \(\beta_y\) are not independent in the classical lamination theory both of them are related to the deflection \(w\).
According to the suggestion of Cheng and Reddy (2001), the complete eigen-relation is given by adding (4.7) with two arbitrarily different multipliers respectively to the 4th and 7th equations of (4.22). To have a definite expression, we select these two multipliers to be $-1/2$ and $1/2$, and the final complete eigen-relation can then be expressed as

$$N_u \xi = \mu \xi, \quad (4.27)$$

where

$$N_u = (L_2 + \frac{1}{2} J_2)^{-1} (L_1 + \frac{1}{2} J_1), \quad (4.28)$$

and

$$L_1 = \begin{bmatrix} 0 & Q \\ R^T - I \end{bmatrix}, \quad L_2 = \begin{bmatrix} R & I \\ T & 0 \end{bmatrix}, \quad \xi = \begin{bmatrix} a \\ b \end{bmatrix}, \quad (4.29)$$

5. REDUCTION FROM UNSYMMETRIC LAMINATES TO SYMMETRIC LAMINATES

In section 4, no symmetry condition has been put on the laminates, whereas in section 3 we require the laminates be symmetric, i.e., the coupling stiffnesses $B_{ij}$ are zero. Hence, if we substitute the requirement $B_{ij} = 0$ into the formalism for the unsymmetric laminates, it is expected to get the formalism for the symmetric laminates. However, because the procedures to establish these two formalisms are different it is really hard to prove their equivalence by just direct substitution. Due to the relative simplicity of symmetric cases, the explicit expressions for the fundamental matrix $N_s$ and its associate eigenvectors have all been found for the symmetric laminates, as those shown in (3.11) and (3.12); whereas those for the symmetric laminates as shown in (4.27) need further numerical implementation. Owing to this slight difference, some features of Stroh formalism for two-dimensional problems cannot directly apply to the formalism for the unsymmetric laminates. If we can successfully reduce the formalism for the unsymmetric laminates to that for the symmetric laminates, it is hopeful that we can find a way to modify the formalism introduced in this paper. Following is the attempt we have done for the reduction.

Substituting $B_{ij} = 0$ into (4.26), we obtain

$$Q_b = R_b = T_b = 0.$$

With this result and the definitions of $Q$, $R$ and $T$ given in (4.23), the eigen-relation (4.27) can now be separated into two parts as

$$\begin{bmatrix} -R_A & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & Q_A \\ T_A^T - I \end{bmatrix} \begin{bmatrix} a_N \\ b_N \end{bmatrix} = \mu \begin{bmatrix} a_N \\ b_N \end{bmatrix}, \quad (5.1)$$

and

$$\begin{bmatrix} -R_D - \frac{1}{2} I_{21} & -I + \frac{1}{2} I_{22} \\ -T_D + \frac{1}{2} I_{11} & -1 + \frac{1}{2} I_{12} \end{bmatrix}^{-1} \begin{bmatrix} 0 & \frac{1}{2} I_{22} \\ \frac{1}{2} I_{21} \end{bmatrix} \begin{bmatrix} a_M \\ b_M \end{bmatrix} = \mu \begin{bmatrix} a_M \\ b_M \end{bmatrix}. \quad (5.2)$$

In the above, (5.1) corresponds to the in-plane problems, and (5.2) is for the plate bending problems. By careful comparison, we see that (5.1) is identical to that for the two-dimensional
problems, but (5.2) is different even we have obtained the identical eigen-relation in (3.7). Because (5.2) and (3.7) are the eigen-relations of the same bending problems, they should be equivalent. If we can prove that they are equivalent, we may improve the formalism for unsymmetric laminates to more Stroh-like formalism and get instant benefit from the experience for two-dimensional problems. Hope this work can be finished in the near future.

6. CONCLUSIONS

Two novel complex variable formalisms for the classical lamination theory are established in this paper. One is for symmetric laminates, the other is for unsymmetric laminates. Their associated eigen-relations are also obtained. Almost all the relations have been purposely arranged to have the same form as those of the corresponding two-dimensional formalism. Due to the similarity, almost all the mathematical techniques developed for two-dimensional problems can lend to the plate bending problems for composite laminates. By simple analogy, many problems which cannot be solved previously, now have the possibility to be solved analytically.

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