Stress Singularities of Multi-Bonded Anisotropic Wedges

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Abstract

By employing the Stroh formalism, a general solution satisfying the basic laws of two-dimensional linear anisotropic elasticity has been written in a complex variable formulation. To study the stress singularity, suitable stress functions have been assumed. The singular order near the multi-bonded wedge apex can then be found by satisfying the boundary conditions. Since there are many material constants and boundary conditions involved, the characteristic equation for the singular order usually becomes cumbersome. It is therefore difficult to get any important parameters to study the failure initiation of the multi-bonded wedges. Through a careful mathematical manipulation, a key matrix $\hat{N}$ that contains the information of material properties and wedge geometries has been found to be a dominant matrix for the determination of the singular order. A closed-form solution for the order of stress singularity is thus written in a simple form. Special cases such as the wedge corners, cracks, interfacial joints or cracks, a crack terminating at the interface, free edges of multiplayer media, etc. can all be studied in a unified manner.

Key Words: Stress singularity, Multi-bonded wedges, Anisotropic materials, Composite materials, Anisotropic elasticity, Stroh formalism, Complex variable method

1. Introduction

The stress singularity generally occurs at the location of discontinuity. The discontinuity may come from geometries, materials or loads, of which the typical examples are, respectively, cracks, multiplayer media, or point forces. Due to the extremely high stresses near the points of discontinuity, failure is usually initiated at such locations. The study of the singularity is generally helpful for the understanding of failure initiation. The nature of stress singularity for the above typical examples has been investigated by many researchers and is illustrated in standard texts such as those by Anderson (1), Jones (2), and Johnson (3). Geometrical discontinuity other than cracks, which has also been studied vastly is the singular stresses at wedge corners. They have been studied for single wedges by Williams (4), England (5), Bogé (6), Stern and Soni (7), Chen (8), etc., and for bonded wedges by Bogé (9–10), Dundurs (11), Hein and Erdogan (12), Theocaris (13), Lin and Hartmann (14), Reedy (15–16), Chen and Nishitani (17), Ding, et al. (18), Berger, et al. (19), and Desmorat and Leckie (20), etc. However, due to the mathematical complexity, most of the studies are restricted to a single material or bi-materials, especially when the wedges are composed of anisotropic elastic materials.
Through the introduction of a key matrix $\tilde{N}$ which includes the information of material properties and wedge angles, we found that each wedge can be represented by its own $\tilde{N}$. Therefore, no matter how many wedges are bonded together, the closed-form solutions for the order of stress singularity can easily be constructed by simple multiplication of $\tilde{N}$ for bonded wedges. Because the final expressions are quite simple and are dominated by the matrix $\tilde{N}$, the results presented in this paper may be useful for the understanding of failure initiation of bonded wedges.

2. Field Equations Near the Wedge Apex

Consider a multi-bonded wedge that is composed of several different anisotropic elastic wedges as shown in Figure 1. If all the wedges satisfy the basic laws for two-dimensional linear anisotropic elasticity, the displacement and stress fields of each wedge can be expressed as

$$
\begin{align*}
\mathbf{u} &= 2 \text{Re}[\mathbf{A}f(z)], \\
\phi &= 2 \text{Re}[\mathbf{B}f(z)],
\end{align*}
$$

(1)

where $\text{Re}$ stands for the real part, $\mathbf{u} = (u_x, u_y, u_z)$ is the displacement vector and $\phi = (\phi_1, \phi_2, \phi_3)$ is the stress function vector which is related to the stresses $\sigma_{ij}$ by

$$
\sigma_{ij} = -\partial \phi_i / \partial x_j, \quad \sigma_{ij} = -\partial \phi_j / \partial x_i.
$$

(2)

$f(z)$ is a function vector composed of three holomorphic complex functions which will be determined through the satisfaction of the boundary conditions. The argument $z$ of each component function is written as $z = x_1 + px_2$ in which $p$ is the material eigenvalue which has been proved to have three pairs of complex conjugates. In the formulation the first three material eigenvalues are arranged to be those having the positive imaginary part. Therefore, each component function of $f(z)$ is a function with different argument $z$ which contains different material eigenvalue $p$ with positive imaginary part. $\mathbf{A} = [a_1, a_2, a_3]$ and $\mathbf{B} = [b_1, b_2, b_3]$ are $3 \times 3$ complex matrices of which $(a_\alpha, b_\alpha), \alpha = 1, 2, 3,$ are the material eigenvectors associated with the first three material eigenvalues and can be determined by the following eigen-relation:

$$
\mathbf{N}z = p^2 z, \quad z = \begin{bmatrix} a \\ b \end{bmatrix},
$$

(3)

where $\mathbf{N}$ is a $6 \times 6$ fundamental matrix and is related to the elastic constants only$^{(2)}$. The surface traction $\mathbf{t}$ at a point on a curve boundary can be calculated according to the Cauchy’s law

$$
\mathbf{t} = \sigma_n n,
$$

where $\mathbf{n}$ is the normal of the boundary. By using the relation given in (2), it can easily be proved that the surface traction is related to the stress function vector by

$$
\mathbf{t} = \partial \mathbf{f} / \partial s,
$$

(4)

where $s$ is the arc length measured along the curved boundary.

To consider the stress singularity at the wedge apex, we assume

$$
f(z) = < z^{-\delta} > \mathbf{g},
$$

(5)

where the angular brackets $<>$ stands for the $3 \times 3$ diagonal matrix, i.e., $< f > = \text{diag}(f_1, f_2, f_3)$ in which three diagonal components are calculated based upon the first three material eigenvalues $p$. $\delta$ is the singular order to be determined from the boundary conditions, and $\mathbf{g}$ is its associated coefficient factor. Since the stress is proportional to the first derivative of the stress function, if $\text{Re}(\delta) > 0$ the stress at the wedge apex will be singular. However, when $\text{Re}(\delta) > 1$ the strain energy of the elastic wedge may become unbounded. Thus, in the following derivation, we will only be interested in the region $0 < \text{Re}(\delta) < 1$.

Substituting (5) into (1), and considering that the singular order $\delta$ may come in pair if they are complex$^{(21)}$, the displacement and stress function vectors of each wedge may be expressed as

$$
\begin{align*}
\mathbf{u} &= \mathbf{A} < z^{-\delta} > \mathbf{g} + \mathbf{A} < z^{-\delta} > \mathbf{h}, \\
\phi &= \mathbf{B} < z^{-\delta} > \mathbf{g} + \mathbf{B} < z^{-\delta} > \mathbf{h},
\end{align*}
$$

(6)

where $\mathbf{g}$ and $\mathbf{h}$ are two complex coefficient vectors to be
determined through the satisfaction of the boundary conditions. If the singular order \( \delta \) is a real value, \( g \) and \( h \) should be complex conjugate to keep the displacements and stresses to be real. Generally, they are not necessary to be complex conjugate. The final real values of the displacements and stresses will come from the superposition with another set of solution whose singular order is \( \delta' \).

For the description of boundary conditions of the wedge problems it is better to use the polar coordinate system \((r, \theta)\), where \( x_1 = r \cos \theta, x_2 = r \sin \theta \). Thus,
\[
z = \hat{r} \hat{t}, \quad \hat{r} = \cos \theta + p \sin \theta.
\]
Substituting (7) into (6) and using the relation (4) for the wedge surfaces \( \theta = \text{constant} \), we have
\[
\begin{align*}
u &= r^{-1-\delta} \left[ A \left< \hat{r} \right>^{-1-\delta} + \nabla_x < \hat{r} \hat{t} > h \right], \\
t &= r^{-1-\delta} \left[ B \left< \hat{r} \right>^{-1-\delta} + \nabla_x < \hat{r} \hat{t} > h \right],
\end{align*}
\]
for \( \theta = \text{constant} \).
\[
(8)
\]
\section{Stress Singularities}

In order to know the value of the singular order \( \delta \), suitable assumptions describing the bonding and surface conditions are needed. In this paper, we assume the wedges are perfectly bonded along each interface between two dissimilar wedges and each wedge occupies the region
\[
\alpha_{k-1} \leq \theta \leq \alpha_k, \quad k = 1, 2, \ldots, n.
\]
Both of the first wedge and the nth wedge are assumed to have one free surface. The traction free boundary condition on the first and the nth wedges as well as the traction and displacement continuity across each interface between two dissimilar wedges can then be written as
\[
\begin{align*}
t_1(\alpha_0) &= 0, & t_n(\alpha_n) &= 0, \\
\n_1(\alpha_k) &= \n_1(\alpha_{k+1}), & \n_1(\alpha_k) &= \n_1(\alpha_{k+1}).
\end{align*}
\]
where the subscript \( k, k = 1, 2, \ldots, n \), are used to denote the quantities pertaining to the \( k \)th wedge. Substituting (8) into the boundary conditions (10), and using the augmented matrix representation, we obtain
\[
\begin{align*}
&\begin{bmatrix} A & \nabla_x \hat{r} \hat{t} \end{bmatrix} \left< \hat{r} \left( \alpha_0 \right) \right>^{1-\delta} = 0, \\
&\begin{bmatrix} 0 & \nabla_x \hat{r} \hat{t} \end{bmatrix} \left< \hat{r} \left( \alpha_n \right) \right>^{1-\delta} = 0,
\end{align*}
\]
\[
(11a)
\]
\[
\begin{align*}
&\begin{bmatrix} A & \nabla_x \hat{r} \hat{t} \end{bmatrix} \left< \hat{r} \left( \alpha_0 \right) \right>^{1-\delta} = 0, \\
&\begin{bmatrix} 0 & \nabla_x \hat{r} \hat{t} \end{bmatrix} \left< \hat{r} \left( \alpha_n \right) \right>^{1-\delta} = 0.
\end{align*}
\]
\[
(11b)
\]
Due to the complexity of equations (11a,b,c), it seems difficult to find a simple solution form to express the singular order. By careful mathematical derivation and the introduction of a key matrix \( \hat{N}_n^{(2)} \), we found that the singular order of the multi-bonded wedges with free-surfaces can be obtained by
\[
\left< \hat{r} \left( \alpha_{j-l} \right) \right>^{1-\delta} \quad \left< \hat{r} \left( \alpha_j \right) \right>^{1-\delta} = 0,
\]
\[
(12)
\]
where \( \hat{N}(\alpha, \beta) \) is defined as
\[
\hat{N}(\alpha, \beta) = \hat{N}(\alpha) \hat{N}^{-1}(\beta),
\]
\[
(13)
\]
It has also been proved that \( \hat{N}_n^{(2)} \)
\[
\begin{bmatrix} A & \nabla_x \hat{r} \hat{t} \end{bmatrix} \left< \hat{r} \left( \alpha \right) \hat{r} \left( \beta \right) \right> = 0,
\]
\[
(14)
\]
The matrix with the subscript \( L \) denotes the lower half part of that matrix. For example, \( \xi_L = b \) where \( \xi \) is defined in (3).

\section{Concluding Remarks}

The explicit closed-form solution shown in (12) is valid for any number of the wedges (for wedge number greater than three, the solutions can be used directly; for one- or two-wedge problems, the solutions should be modified slightly \( \hat{N}_n^{(2)} \)). Due to the consideration of the stress singularity, the stress function has been chosen to have the form \( \hat{z}^{1-\delta} \). From the eigen-relation (3), we observe that this function form is directly related to the key matrix \( \hat{N}_n^{1-\delta}(\theta) \). Since each wedge is bounded by two wedge surfaces \( \theta = \alpha_{j-l} \) and \( \theta = \alpha_j \), it is reasonable to predict that the singular order should be related to \( \hat{N}_n^{1-\delta}(\alpha_{j-l}, \alpha_j) \). Furthermore, the solution shows that if a wedge is bonded with the other wedge,
the final results should be related to the multiplication of $\tilde{N}^{-\delta} (\alpha_{j-1}, \alpha_{j})$ of the bonded wedges. If the wedge has one free surface, the singular order is related to the lower half of its key matrix. Therefore, through the introduction of the key matrix $\tilde{N}$, one may write down the solutions for the unsolved problems without any detailed derivation. It is very convenient for the researchers to understand the effects of the material properties, wedge angles, and the combination of the wedges on the stress singularities of the multi-bonded wedges.

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