Aeroelastic Divergence of Composite Cantilever Swept Wing

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ABSTRACT

By assuming that the wing chordwise section is rigid and treating the wing as a composite sandwich plate, a composite cantilever swept wing model is established in this paper. Since all force terms are retained without expressing them in terms of deformations, the final equilibrium equations and boundary conditions become compact and easily understandable equations. This helps us to extend the formulation to more complicated and realistic situations such as the consideration of the fiber orientations, ply stacking sequences, stringers, spars, ribs and the shapes of airfoils. To study the aeroelastic divergence problems, we employ this formulation to represent the composite wing structure and use aerodynamic strip theory to compute the lift loads. The system of equations is then solved by using the technique of Laplace integral transform. Illustrative numerical examples are also given to show the effects of aspect ratios, swept angles and transverse shear strains on the divergence dynamic pressures and the lift loads redistribution.

Keywords: aeroelasticity, divergence speed, composite, swept wing, sandwich plate.

1. INTRODUCTION

Due to the characteristics of designability and light-weight of the composites, several aircraft designs which are thought to be impossible for the metallic materials become possible for the composite materials. One of the examples is the aeroelastic divergence for swept-forward wings. It was well known that bending deflections have a destabilizing effect on the swept-forward wings and hence the swept-forward wing aircraft as a possible option was completely eliminated for a long time. However, some of the research results indicates that swept-forward wing aircraft may be a better choice than swept-back wing such as forward swept wings experience stall at higher angles of attack than swept-back wings. From a design optimization standpoint, it is always interested to have the choices not confined to swept-back configuration. Therefore, when studies by Krone revealed (using a numerical approach) that a composite swept-forward wing can be aeroelastically tailored to overcome the divergence problem, a considerable amount of works were done in the following years. For example, Weisshaar proposed an analytical model for the aeroelastic tailoring and concluded that the aeroelastic divergence may be influenced by the elastic coupling between wing bending and torsional deformations introduced by laminated composite construction. Oyibo proposed three generic stiffness variables and the fiber orientation angle for the optimum aeroelastic characterics. A series works done by Librescu and his co-workers considered the warping restraint effects, and the effects of transverse shear strains. All these results support that a composite swept-forward wing can be tailored to overcome this adverse instability phenomenon.

When the analytical model was first proposed to consider the aeroelastic instability, several assumptions were given to simplify the mathematical formulation. Later on, several refined models were proposed in order to be closer to the real wing structures. Hence, the mathematical formulations become complicated and it becomes difficult to catch the physical indication from the complicated equations. In this paper, a relative simple and compact mathematical formulation is derived by using the mechanical model for the composite sandwich plates, and can be proved to be equivalent to that derived in the literature. Due to its simplicity, the physical meaning of each equation is clear and hence is easy to be used to consider more complicated structures such as nonuniform stiffened composite multicell wings. Moreover, the important factors such as the warping restraint, transverse shear strain, shape of airfoil, aspect ratio, swept angle, fiber orientation, ply stacking sequence, stringers and spars, can all be included in the formulation without adding too much complexity.
2. COMPOSITE CANTILEVER SWEPT WINGS

Due to the closely spaced stringers and the transverse stiffening members like wing spars and ribs, in aircraft analysis it is usually assumed that the wing chordwise section is rigid. Moreover, by treating wing skins, stringers, and the spar flanges as the sandwich faces, and wing spar webs and ribs as the sandwich cores, the stiffened composite multicell wing structures may be modelled as a composite sandwich plate.

By adopting the Mindlin-Reissner assumptions for the composite sandwich plates, the displacement field \((u, v, w)\) of the wing structures is assumed to be a linear function of \(z\) where \(z\) is the coordinate in the thickness direction, i.e.,

\[
u = u_0 + z\beta_x, \quad v = v_0 + z\beta_y, \quad w = w_0.\]

\(u_0, v_0\) and \(w_0\) are the mid-plane displacements in the \(x, y\) and \(z\) directions; \(\beta_x\) and \(\beta_y\) are, respectively, the rotation angles with respect to \(x\) and \(y\) axes. Moreover, \(u_0, v_0, w_0, \beta_x, \beta_y\) are all functions of \(x\) and \(y\) only. The rotation angles \(\beta_x\) and \(\beta_y\) are related to the transverse shear strains \(\gamma_{xz}\) and \(\gamma_{yz}\) by

\[
\beta_x = \gamma_{xz} - \frac{\partial w}{\partial x}, \quad \beta_y = \gamma_{yz} - \frac{\partial w}{\partial y}.\]

Consistent with the chordwise-rigid postulation, the mid-plane displacement and rotation angles shown in Eq.(1) may be further assumed to be

\[
u_0 = 0, \quad v_0 = v_0(y), \quad w_0 = w_f(y) - x\theta(y), \quad \beta_x = \theta(y), \quad \beta_y = \beta_f(y) + x\beta_r(y),\]

where \(w_f(y)\) denotes the deflection measured at the line of flexural center, which is now selected as the reference axis shown in Figure 1; \(\theta(y)\) is the twist around the flexural axis. The equivalence of \(\beta_x\) and \(\theta\) is due to the chordwise-rigid assumption which leads to \(\gamma_{xz} = 0\). However, in spanwise direction the transverse shear deformation cannot be neglected for the thick plates. Hence, two extra functions \(\beta_f\) and \(\beta_r\) are needed for the representation of \(\beta_y\), where \(\beta_f\) denotes the rotation angle measured at the flexural axis and \(\beta_r\) stands for the rate of angle change in the \(x\)-direction. By the assumption given in (3), the mid-plane strains and curvatures become

\[
\epsilon_{xx} = 0, \quad \epsilon_{xy} = v_0', \quad \gamma_{yz} = 0, \quad \kappa_x = 0, \quad \kappa_y = \beta_f' + x\beta_r', \quad \kappa_{xy} = \theta' + \beta_r'.\]

\[\text{Figure 1. Geometry of the composite cantilever swept wing}\]
In order to obtain the equations governing the static equilibrium of the composite wing and the associated boundary conditions, the usual way found in the literature is the employment of Hamilton’s variational principle. A direct method by integrating the equilibrium equations of the composite sandwich plates with respect to \( x \)-axis is proposed by Hwu and Tsai. Here, we follow the Hamilton’s variational principle but retain the force terms (without expressing them in terms of displacements, which may make our formulation complicated) in order to have a clear and compact mathematical formulation. Toward this end, the total potential energy \( \Pi \) of the composite wings may be expressed as

\[
\Pi = U - W
\]  

where the strain energy \( U \) is

\[
U = \int \int (N_x \varepsilon_{x} + Q_y \gamma_{x} + M_y \kappa_{x} + M_w \kappa_{y}) \, dxdy
\]  

and the work \( W \) done by the external force is

\[
W = \int \int (p_x v_0 + p m_x + m \beta_x + m \beta_y) \, dxdy
\]  

\[+ \int (N_x v_0 + Q_y w_0 + M_y \beta_x + M_w \beta_y) \, dx
\]  

\((N_x, Q_y)\) and \((M_y, M_w)\) are the stress resultants and bending moments. One may refer to [14] for their relations with the lamina stresses and strains. The overbar means the prescribed values at the boundaries. \( p_x, p \) and \( m_x, m_y \) are the total distributed loads and moments applied on the upper and lower surfaces of the sandwich plates. According to the postulation given in (3), the basic functions describing the deformation of the composite wing structures become \( f \), \( \phi \), \( w \), \( \theta \), \( \beta \), and \( \beta \). Therefore the stationary condition, \( \delta \Pi = 0 \), is considered by taking the variation with respect to these five basic functions. By performing the integration by parts whenever possible and collecting terms and setting the coefficients of \( \delta v, \delta w, \delta \theta, \delta \beta, \) and \( \delta \beta \) to be zero, we may obtain the equilibrium equations and boundary conditions. For a composite wing structure with the spanwise loads as well as the distributed moments neglected (i.e., \( \bar{p}_y = \bar{N}_y = 0 \) and \( \bar{m}_y = \bar{M}_y = 0 \)), the equilibrium equations and boundary conditions can be further reduced to,

\[
\frac{d\bar{Q}_y}{dy} = -\bar{p}, \quad \frac{d(M_y - \bar{Q}_y)}{dy} = \bar{p}, \quad \frac{d\bar{M}_y}{dy} = \bar{Q}_y, \quad -\bar{M}_y + \frac{d\bar{M}_y}{dy} = \bar{Q}_y,
\]  

and

\[
w_f = \theta = \beta_f = \beta_f = 0, \quad \text{at} \quad y = 0 \text{ (wing root)},
\]  

\[
\bar{Q}_y = \bar{M}_y, \quad \bar{Q}_y = 0, \quad \text{at} \quad y = l \text{ (wing tip)}.
\]  

Here, the tilde \(~\) denotes integration with respect to \( x \), the superscript \(*\) denotes multiplication by \( x \). For example,

\[
\bar{D}_{22} = \int_{c_{i}}^{c_{f}} D_{22} \, dx, \quad \bar{D}_{22} = \int_{c_{i}}^{c_{f}} D_{22} \, dx \quad \bar{D}_{22} = \int_{c_{i}}^{c_{f}} D_{22} \, x^2 \, dx, \ldots, \text{etc.}
\]  

The lower and upper limits \(-c_{i}\) and \( c_{f}\) denote, respectively, the location of leading and trailing edges. By a way similar to the classical lamination theory, the relations between \((\bar{Q}_y, \bar{M}_y, \bar{Q}_y, \bar{M}_y)\) and \((w_f, \theta, \beta_f, \beta_f)\) can be written in matrix notation as

\[
F = K_x \Delta + K_x \Delta',
\]  

where the strain energy \( U \) is

\[
U = \int \int (N_x \varepsilon_{x} + Q_y \gamma_{x} + M_y \kappa_{x} + M_w \kappa_{y}) \, dxdy
\]

and the work \( W \) done by the external force is

\[
W = \int \int (p_x v_0 + p m_x + m \beta_x + m \beta_y) \, dxdy
\]

\[+ \int (N_x v_0 + Q_y w_0 + M_y \beta_x + M_w \beta_y) \, dx
\]

\((N_x, Q_y)\) and \((M_y, M_w)\) are the stress resultants and bending moments. One may refer to [14] for their relations with the lamina stresses and strains. The overbar means the prescribed values at the boundaries. \( p_x, p \) and \( m_x, m_y \) are the total distributed loads and moments applied on the upper and lower surfaces of the sandwich plates. According to the postulation given in (3), the basic functions describing the deformation of the composite wing structures become \( f \), \( \phi \), \( w \), \( \theta \), \( \beta \), and \( \beta \). Therefore the stationary condition, \( \delta \Pi = 0 \), is considered by taking the variation with respect to these five basic functions. By performing the integration by parts whenever possible and collecting terms and setting the coefficients of \( \delta v, \delta w, \delta \theta, \delta \beta, \) and \( \delta \beta \) to be zero, we may obtain the equilibrium equations and boundary conditions. For a composite wing structure with the spanwise loads as well as the distributed moments neglected (i.e., \( \bar{p}_y = \bar{N}_y = 0 \) and \( \bar{m}_y = \bar{M}_y = 0 \)), the equilibrium equations and boundary conditions can be further reduced to,

\[
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\]  

and

\[
w_f = \theta = \beta_f = \beta_f = 0, \quad \text{at} \quad y = 0 \text{ (wing root)},
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\]  

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\[
\bar{D}_{22} = \int_{c_{i}}^{c_{f}} D_{22} \, dx, \quad \bar{D}_{22} = \int_{c_{i}}^{c_{f}} D_{22} \, dx \quad \bar{D}_{22} = \int_{c_{i}}^{c_{f}} D_{22} \, x^2 \, dx, \ldots, \text{etc.}
\]  

The lower and upper limits \(-c_{i}\) and \( c_{f}\) denote, respectively, the location of leading and trailing edges. By a way similar to the classical lamination theory, the relations between \((\bar{Q}_y, \bar{M}_y, \bar{Q}_y, \bar{M}_y)\) and \((w_f, \theta, \beta_f, \beta_f)\) can be written in matrix notation as

\[
F = K_x \Delta + K_x \Delta',
\]
where

$$\mathbf{F} = \begin{bmatrix} \tilde{Q}_y \\ \tilde{M}_y \\ Q'_y \\ M'_y \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} w_f \\ \theta \\ \beta_f \end{bmatrix}$$

$$\mathbf{K}_0 = \begin{bmatrix} 0 & 0 & \tilde{A}_{44} & \tilde{A}_{44} \\ 0 & 0 & \tilde{D}_{26} & 0 \\ 0 & 0 & \tilde{D}_{66}^{*} & 0 \\ 0 & 0 & \tilde{D}_{26}^{*} & 0 \end{bmatrix}$$

$$\mathbf{K}_1 = \begin{bmatrix} \tilde{A}_{44} - \tilde{A}_{44}^{*} & 0 & 0 \\ 0 & \tilde{D}_{26} & \tilde{D}_{22} & \tilde{D}_{22}^{*} \\ 0 & \tilde{D}_{66} & \tilde{D}_{26} & \tilde{D}_{26}^{*} \\ 0 & \tilde{D}_{26}^{*} & \tilde{D}_{22}^{*} & \tilde{D}_{22}^{**} \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} \tilde{Q}_y \\ \tilde{M}_y \\ Q'_y \\ M'_y \end{bmatrix}$$

The system of equilibrium equations (8) with boundary conditions (9) can then be solved by using the technique of Laplace integral transform. This technique transforms (8) to a linear algebraic system of equations as, in matrix notation,

$$\mathbf{S}\hat{\mathbf{F}}(s) = \mathbf{J}\mathbf{F}(0) + \mathbf{P}(s),$$

(14)

where

$$\mathbf{S} = \begin{bmatrix} s & 0 & 0 & 0 \\ 0 & 0 & s & -s \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{P}(s) = \begin{bmatrix} -\hat{p}(s) \\ \hat{p}'(s) \end{bmatrix}$$

(15)

The overhat ^ denotes the Laplace transform, e.g., $\hat{\tilde{Q}}_y(s) = L[\tilde{Q}_y(y)]$. By (11), we have $\mathbf{F}(0) = \mathbf{K}_0\Sigma(0) + \mathbf{K}_1\Sigma'(0)$. The boundary condition (9), means that $\Sigma(0) = 0$. Hence, $\mathbf{F}(0) = \mathbf{K}_0\Sigma'(0)$. The Laplace transform of (11) plus the boundary condition, $\Sigma(0) = 0$, leads to $\hat{\mathbf{F}}(s) = (\mathbf{K}_0 + s\mathbf{K}_1)\hat{\mathbf{\Sigma}}(s)$. Substituting these results into (14), we get the governing equations for the composite wing structures in the transformed domain of $w_f$, $\theta$, $\beta_f$ and $\beta$, as

$$\mathbf{S}(\mathbf{K}_0 + s\mathbf{K}_1)\hat{\mathbf{\Sigma}}(s) = \mathbf{J}\mathbf{K}_0\hat{\mathbf{\Sigma}}'(0) + \mathbf{P}(s),$$

(16)

and the boundary conditions remained to be satisfied are those given in (9)2. By using the matrix notation and the relation given in (11), the boundary conditions (9)2 can be rewritten as

$$\mathbf{J}\mathbf{F}(l) = \mathbf{J}[\mathbf{K}_0\Sigma(l) + \mathbf{K}_1\Sigma'(l)] = 0.$$
3. AEROELASTIC DIVERGENCE

To study the aeroelastic divergence phenomena, the applied forces \( \bar{p} \) and \(- \bar{p}'\) can be considered as the lift and the aerodynamic nose-up torsional moment (per unit length), respectively. For the static case considered here, \( \bar{p}(y) \) and \(- \bar{p}'(y) \) are

\[
\bar{p}(y) = a q_v c \theta_{\text{eff}}(y),
\]

\[
- \bar{p}'(y) = a q_v c \theta_{\text{eff}}(y) + q_c^2 C_{m,\text{ac}},
\]

where

\[
\theta_{\text{eff}}(y) = \theta_0 + \theta(y) - w(y) \tan \lambda.
\]

\( a \) is the lift curve slope coefficient; \( q_v \) denotes the dynamic pressure component normal to the leading edge; \( c \) is the wing chord length and \( e \) is the distance between the lines of aerodynamic and flexural centers, both of which are considered to be constant in this paper; \( \theta_0 \) denotes the angle of attack corresponding to the rigid wing assumption; \( \lambda \) is the angle of sweep (positive for swept-back and negative for swept-forward); \( C_{m,\text{ac}} \) is the pitching moment coefficient about the aerodynamic center, which is a constant with respect to the angle of attack. The related equations about \( q_v \) and \( a \) are

\[
q_v = \frac{1}{2} \rho V_a^2 = q \cos^2 \lambda, \quad a = \frac{dC_L}{d\alpha} = a_0 \frac{AR}{AR + 4 \cos \lambda},
\]

where \( \rho \) is the density of the airflow; \( V_a \) is the airflow velocity component normal to the leading edge; \( q \) is the dynamic pressure; \( C_L \) is the lift coefficient; \( a_0 \) is the corresponding two-dimensional lift-curve slope; \( AR \) is the wing aspect ratio defined as \( AR = (2l)^2 / S = 2l / c \) where \( 2l \) is the wingspan measured as the distance between the two wingtips and \( S(=2cl) \) in the case that \( c \) is independent of \( y \) is the total area of the wing in the planform (x-y plane) view.

Taking the Laplace transform of Eq.(18) and using the matrix notation defined in (15), we get

\[
P(s) = \frac{1}{s} P_0 + A \hat{\Lambda}(s),
\]

where

\[
P_0 = -q_v c \begin{bmatrix} a \theta_0 \\ a e \theta_0 + c C_{m,\text{ac}} \\ 0 \\ 0 \end{bmatrix}, \quad \Lambda = a q_v c \begin{bmatrix} s \tan \lambda - 1 & 0 & 0 \\ s \tan \lambda - e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Substituting (21) into (16), and implementing the inverse Laplace transform, we get

\[
\Lambda(y) = K_s(y) J K_s \Lambda'(0) + \left[ \int_0^y K_s(y) dy \right] P_0,
\]

where \( K_s(y) \) is the inverse Laplace transform of \( \hat{K}_s(s) \), and

\[
\hat{K}_s(s) = [S(K_0 + sK_s) - \Lambda]^{-1}.
\]

The unknown vector \( \Lambda'(0) \) can then be determined by substituting (23) into the boundary condition (17), which leads to
\[ J[K_yK_z(l)JK_1 + K_yK_z'(l)JK_1]A'(0) = -J[K_y(\int_y(y)dy)] + K_yK_z(l)P_0. \]  

The solution of \( A'(0) \) can, therefore, be uniquely determined from (25). The four basic functions \( \Lambda(y) = (w, \theta, \beta_y, \beta_\gamma) \) are then determined by (23). The resultant forces and bending moments \( F(y) = (\tilde{Q}_y, \tilde{M}_y, \tilde{M}_\gamma, \tilde{Q}_\gamma, \tilde{M}_\gamma) \) can also be found by the relation given in (11). However, in some conditions the determinant of the coefficient matrix of \( A'(0) \) may become zero, which will lead to the results that \( A'(0) \) is infinite. In other words, under a certain condition the deflection of the wing structures may become infinite, which is the situation of aeroelastic divergence discussed in this paper. The lowest value of \( q \) for which the determinant vanishes corresponds to the critical (divergence) dynamic pressure \( q_d \), and its associated velocity \( V_d \) is called divergence speed.

4. NUMERICAL RESULTS AND DISCUSSIONS

Although the thickness of the wing structure is generally not very thin, most of the study on aeroelastic divergence problems did not consider the effect of transverse shear strain. Karpouzian and Librescu\(^8\) included the transverse shear strain effect in their comprehensive model for anisotropic composite aircraft wing, and stated its importance on the aeroelastic divergence problems. However, only a flat rectangular isotropic wing model was performed in their illustrating examples. Before illustrating more general and realistic examples, we first check our formulation by comparing our solutions with this special case. The results show that they agree well with the existing numerical solutions\(^14\).

After verifying our results by using the flat rectangular isotropic wing model, we consider a composite wing structure with NACA 2412 airfoil. Its associated aerodynamic data are \( a_0 = 5.73 \text{ rad}^{-1}, C_{\mu,\text{aw}} = -0.04, e/c = 0.257, \) and \( \theta_0 = S^0 \). The wing chordwise length \( e = 1 \text{m} \). The wing skin is made of graphite/epoxy fiber-reinforced composite whose mechanical properties are: \( E_{11} = 200 \text{ GPa}, E_{22} = 5 \text{ GPa}, v_{12} = 0.25, G_{12} = 2.5 \text{ GPa} \) and ply thickness \( t = 0.25 \text{ mm} \). The ply stacking sequence is \([90/-45/45/0]\) for the upper skin and \([0/45/-45/90]\) for the lower skin. Figures 2 and 3 show the effects of aspect ratio and swept angle on the divergence dynamic pressure. The lift redistribution due to the distortion may be reflected by the angle of attack \( \theta_{\text{eff}} \) given in (19). Figures 4, 5 and 6, show, respectively, the effects of transverse shear
strain, aspect ratio and swept angle on the angle of attack. Basically, the trend shown in these figures agree with those presented in the literature for the special cases \cite{3-10}. Detailed discussion and implication of these numerical results can be found in our other parallel paper \cite{14}.

**Figure 3.** The effect of aspect ratio on divergence dynamic pressure.

**Figure 4.** Spanwise distribution of the effective angle of attack with the variation of transverse shear modulus.

\( (AR = 6, \lambda = -30^\circ, q = 30 KPa) \)
Figure 5. Spanwise distribution of the effective angle of attack with the variation of swept angle. ($AR = 8$, $q = 8KPa$)

Figure 6. Spanwise distribution of the effective angle of attack with the variation of aspect ratio. ($\lambda = \pm 30^\circ$, $q = 10KPa$)
5. CONCLUDING REMARKS

By retaining the force terms in the derivation, the final equilibrium equations and the associated boundary conditions of the composite cantilever swept wings become compact and relatively simple mathematical expressions. With these simple equations, more complicated and realistic situations of the composite wing structures can be easily extended. Moreover, the physical meaning of the aeroelastic stability problems can also be caught more easily than those presented in the literature. To show the generality of our results, under the consideration of transverse shear strain a composite wing with NACA 2412 airfoil is illustrated in this paper. The results show that they preserve the same trend as those of the flat wings with the transverse shear strain neglected, except that their values will be influenced by the shape of the airfoil and the transverse shear strain.

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