Evaluation of Fracture Parameters by the Response Away from the Cracks

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Abstract

By using the Stroh formalism for two-dimensional anisotropic elasticity and the proper boundary element formulation for the internal straight crack, a closed-form solution for the evaluation of the stress intensity factors and the energy release rates are provided in this paper. This formula is based upon the displacements and tractions away from the crack. Since the input data can be far removed from the crack tips, the results obtained here are accurate and the method is more efficient than the conventional near tip approach. The formula presented here can be applied to any kind of anisotropic materials. The two-dimensional problems discussed here include the generalized plane stress, the generalized plane strain and the anti-plane problems.

1. Introduction

It is well known that the stresses near the crack tip around the homogeneous materials exhibit a square-root singularity. To denote the stress intensity induced by the crack and to predict the propagation of the crack, the fracture parameters like the stress intensity factors (SIF) and energy release rates (ERR) are usually used [1]. Although several analytical solutions for SIF and ERR can be found in the literature such as [2], they are limited to a few idealized unbounded-bodies. For practical problems involving finite geometry and complex loading, numerical methods such as the finite element and boundary element methods are usually employed [3]. Through these methods, the data near the crack tips are generally required. However, due to the singularity characteristics, the stresses and strains near the crack tips obtained by the numerical methods are usually unreliable. To overcome this difficult, a famous path independent J-integral has been proposed by Rice about thirty years ago, by which SIF and ERR can be calculated by using data far removed from the crack tips [4]. However, the J-integral cannot be applied to problems where mixed-mode SIFs are present since the value of the integral only gives the sum of SIFs. Moreover, the path for the J-integral should start from and end to the crack surfaces. Several other alternative path-independent integrals were proposed later on to solve this problem [5-7].

Recently, my co-workers and I developed a special boundary element method dealing with the problems with multi-holes, cracks and inclusions [8-10]. Through
The closed form solutions of SIF and ERR presented in this paper are valid for the internal crack embedded in the general two-dimensional anisotropic elastic solids. It is common to employ the homogenization concept to the measuring of the fracture properties of composite materials. Therefore, from macroscopic viewpoint, our results can be applied to the most complicated fibrous composites or particulate composites. It is also not difficult to extend our result to the anisotropic piezoelectric materials.

2. Stress Intensity Factors

The stress intensity factors for a crack embedded in a homogeneous material are defined as [14]

\[
\mathbf{K} = \begin{bmatrix} K_{II} \\ K_{I} \\ K_{III} \end{bmatrix} = \lim_{r \to 0} \sqrt{2\pi r} \begin{bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{bmatrix}
\]

(1)

where \( K_I, K_{II}, K_{III} \) denote, respectively, the mode I (opening mode), II (shearing mode) and III (tearing mode) stress intensity factors of the crack; \( \sigma_y \) stand for the stresses and \( r \) is the distance ahead of the crack tip. Through the use of the boundary element formulation developed by Hwu and Yen [8] and the Stroh formalism for the anisotropic elasticity [12], a closed-form solution for the stress intensity factors has been derived in [10] as

\[
\mathbf{K} = \sum_{n=1}^{N} \{ [\mathbf{R}^T \mathbf{G}_{n}^{*} + \mathbf{T} \mathbf{G}_{n}^{*}^T] \mathbf{t}_n - [\mathbf{R}^T \mathbf{Y}_{n}^{*} + \mathbf{T} \mathbf{Y}_{n}^{*^T}] \mathbf{u}_n \} \quad (2a)
\]

where \( N \) is the total number of the remote boundary points to be input for the calculation of the stress intensity factor. The superscript \( T \) denotes the transpose of a matrix, and

\[
\mathbf{R} = \begin{bmatrix} C_{16} & C_{12} & C_{14} \\ C_{66} & C_{62} & C_{64} \\ C_{56} & C_{52} & C_{54} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} C_{66} & C_{62} & C_{64} \\ C_{26} & C_{22} & C_{24} \\ C_{46} & C_{42} & C_{44} \end{bmatrix} \quad (2b)
\]

\( C_y \) are the elastic constants of the materials. \( \mathbf{G}_{ni}^{*} \) and \( \mathbf{Y}_{ni}^{*} \), \( i=1,2 \), can be calculated by

\[
\mathbf{G}_{ni}^{*} = \frac{1}{2} \int_{-1}^{1} \mathbf{U}_i^*(1+\xi) l_{n,n-1} d\xi + \frac{1}{2} \int_{1}^{1} \mathbf{U}_i^*(1-\xi) l_{n,n+1} d\xi
\]

\[
\mathbf{Y}_{ni}^{*} = \frac{1}{2} \int_{-1}^{1} \mathbf{T}_i^*(1+\xi) l_{n,n-1} d\xi + \frac{1}{2} \int_{1}^{1} \mathbf{T}_i^*(1-\xi) l_{n,n+1} d\xi, \quad i=1,2 \quad (2c)
\]

where \( l_{n,n-1} \) denotes the length between nodes \( n \) and \( n-1 \), and \( l_{n,n+1} \) is the length between nodes \( n \) and \( n+1 \), and \( \mathbf{U}_i^* \) and \( \mathbf{T}_i^* \) are defined as

\[
\mathbf{U}_i^* = \frac{1}{2\sqrt{\pi a}} \text{Im} \{ \mathbf{A} \ll \frac{1}{1-\zeta} \gg \mathbf{A}^T - \mathbf{A} \ll \frac{\zeta}{1-\zeta} \gg \mathbf{B}^{-1} \mathbf{B}^T \},
\]
\[ U_z = \frac{1}{2\sqrt{\pi a}} \text{Im} \{ A \ll p_\alpha \gg A^T - \sum_{k=1}^{3} A \ll p_k \zeta_k \gg B^{-1} \text{Bi}_k \bar{A}^T \}, \]

\[ T_1 = \frac{1}{2\sqrt{\pi a}} \text{Im} \{ B \ll p_\alpha \zeta_\alpha^2 (s_1 + p_\alpha s_2) \gg (A^T - B^{-1} \bar{B} \bar{A}^T) \}, \]

\[ T_2 = \frac{1}{2\sqrt{\pi a}} \text{Im} \{ B \ll p_\alpha \zeta_\alpha^2 (s_1 + p_\alpha s_2) \gg A^T - \sum_{k=1}^{3} B \ll p_k \zeta_k^2 (s_1 + p_\alpha s_2) \gg B^{-1} \text{Bi}_k \bar{A}^T \}, \]

(2d)

in which \( \text{Im} \) denotes the imaginary part of a complex number, and the overbar stands for the complex conjugate. The angular bracket \( \ll \gg \) denotes the diagonal matrix in which the diagonal term varies according to the subscript \( \alpha \). \( a \) is the length of the crack. \( p_\alpha \) and \( (A, B) \) are, respectively, the material eigenvalues and eigenvector matrices which are complex numbers and can be determined by the elastic constants \( C_{ij} \) [12]. \( s_1 \) and \( s_2 \) are, respectively, the direction cosine and sine of the line connecting nodes \( n \) and \( n-1 \) (or nodes \( n \) and \( n+1 \)). \( \zeta_\alpha \) is related to the integration variable \( \zeta \) by

\[ \zeta_\alpha = \frac{1}{a} \{ z_\alpha - \sqrt{z_\alpha^2 - a^2} \} \]

(2e)

where

\[ z_\alpha = \frac{1}{2} [(1 - \xi)(x^{(1)} + p_\alpha y^{(1)}) + (1 + \xi)(x^{(2)} + p_\alpha y^{(2)})] \]

(2f)

and \( (x^{(0)}, y^{(0)}) \) are the locations of the nodal points. For the segment connecting nodes \( n \) and \( n-1 \), the superscript (1) denotes the node \( n-1 \) while (2) denotes the node \( n \). Similarly, for the segment connecting nodes \( n \) and \( n+1 \), the superscript (1) denotes node \( n \) while (2) denotes the node \( n+1 \). Since the calculation of \( s_1, s_2, (x^{(1)}, y^{(1)}) \) and \( (x^{(2)}, y^{(2)}) \) depends on the segment to be integrated, the values of \( U_z \) and \( T_1 \) calculated from (2d) may be different for the first integral term and the second integral term of (2c)1 and (2c)2.

Note that eqn. (2) provides a direct method for evaluating the stress intensity factor if the remote displacements \( u_n \) and tractions \( t_n \) are known on some closed contour containing the crack. The values of \( u_n \) and \( t_n \) on the remote closed contour may be supplied by any method like the finite element method or the experimental measurement. This is very different from the conventional computation of the stress intensity factors since they are usually obtained from the data near the crack tip. Even by the well-known path-independent J-integral, we still used a path starting and ending at the crack surfaces. While by eqn. (2), all the data used to calculate \( K \) are from the remote boundary. Using the conventional finite element to find a convergent solution for the stress intensity factors usually requires fine meshes near the crack tips. This is not only time-consuming but also inaccurate. All these defects have been overcome by the present formula since fine meshes near the crack tips are not necessary to get a convergence value of \( K \).
3. Energy Release Rates

The relation between the stress intensity factors and the energy release rates for the general anisotropic elastic materials are [15]

\[ G = \mathbf{K}^T \mathbf{L}^{-1} \mathbf{K} \]  

(3a)

where \( \mathbf{L} \) is a \( 3 \times 3 \) real symmetric matrix composed of the elasticity constants, and is defined as

\[ \mathbf{L} = -2 \mathbf{BB}^T. \]  

(3b)

It can be proved that \( \mathbf{L} \) is positive definite, if the strain energy is positive [16]. Since the eigenvector matrix \( \mathbf{B} \) does not exit for the materials with repeated eigenvalues \( p_a \), this definition is not valid for degenerate materials for which isotropic material is a special case. Alternative definition for \( \mathbf{L} \) of degenerate materials can be found in [17] or \( \mathbf{L} \) can be calculated by the integral formalism proposed by Barnett and Lothe [18]. But the most convenient formula is the explicit expression derived by Wei and Ting [19]. For monoclinic materials with the symmetry plane at \( x_3 = 0 \), the explicit expression of \( \mathbf{L} \) is [12]

\[
\mathbf{L} = \frac{s^2}{s_{11} g^2} \begin{bmatrix}
    e & -d & 0 \\
    -d & b & 0 \\
    0 & 0 & \mu s_{11} g^2 s^{-2}
\end{bmatrix}
\]  

(4a)

where

\[
a + ib = p_1 + p_2, \quad c + id = p_1 p_2, \quad e = ad - bc, \quad b > 0,
\]

\[
g = (s_{12}' / s_{11}') - c > 0, \quad 1 > s = g(b - d^2)^{-1/2} > 0.
\]

\( s_{ij}' \) are the reduced elastic compliances related to the elastic compliances \( s_{ij} \) by

\[
s_{ij}' = s_{ij} - \frac{s_{13}' s_{3j}}{s_{33}}.
\]

(4c)

\( \mu \) is defined as

\[
\mu = \{(s_{44}' s_{55}' - s_{45}' s_{54}')^{-1/2} = \{C_{44} C_{55} - C_{45}^2\}^{1/2}
\]

(4d)

which is the shear modulus when the material is isotropic.

The expression given in (4) can be further simplified for the orthotropic materials [20] and may also be expressed by using the engineering constants [15]. They are

\[
L_{11} = \alpha_1 \kappa_1 E_1, \quad L_{22} = \alpha_2 \kappa_2 E_2, \quad L_{33} = \sqrt{G_{23} G_{31}}
\]

(4e)

and all the other components of \( L_{ij} \) are equal to zero. In (4e), \( E_1 \) and \( E_2 \) are the Young's moduli in 1 and 2 directions, respectively. \( G_{55} \) and \( G_{31} \) are the shear moduli in the 2-3 and 3-1 planes, respectively. \( \alpha_i \) and \( \kappa_i \) are nondimensionless factors related to the Poisson's ratios, Young's moduli and Shear moduli [15]. It should be noted that the explicit expressions shown in (4) is also applicable to the degenerate materials.

4. Numerical Examples

To show that the proposed formulae obtained in (2) for SIF and (3) for ERR are
accurate and efficient, two representative examples are illustrated in this section.

**Example 1: A square laminate containing an inclined crack**
Consider a $[45^\circ/-45^\circ]_s$ laminate containing an inclined crack subjected to uniform loading ($\sigma^{o}_{12}, \sigma^{o}_{22}, \sigma^{o}_{32}$) at the upper and lower surfaces of the laminate (see Figure 1). The mechanical properties of each lamina is $E_1=114.8$ GPa, $E_2=11.72$ GPa, $G_{12}=0.65$ GPa, $\nu_{12}=0.21$. The stress intensity factors and energy release rate of the crack are calculated by using the formulae proposed in eqns.(1), (2) and (3). The results presented in Figure 2 show that the formulae proposed in eqns.(2) and (3) are really more efficient than the conventional near tip solution.

Figure 1: A square anisotropic plate containing an inclined crack
($\sigma^{o}_{12}=\sigma^{o}_{22}=\sigma^{o}_{32}=1$ MPa, $W=0.3$ m, $a=0.06$ m, $x/W=y/W=0.67$, $\theta=30^\circ$)

Figure 2: The left tip stress intensity factors and energy release rate of a crack in a square laminate

**Example 2: A semi-circular isotropic plate containing a horizontal crack**
To show that our formulae can also be applied to the degenerate materials, we
now consider a semi-circular plate made of the isotropic materials whose properties are $E=206.8$ GPa, $G=80.16$ GPa, $\nu_{12}=0.29$ (see Figure 3). The results are shown in Figure 4 which also shows that our formulae are more efficient than the conventional near tip solution.

![Figure 3: A semi-circular isotropic plate containing a horizontal crack](image)

$\sigma_{12}^{0}=\sigma_{22}^{0}=\sigma_{32}^{0}=1$ MPa, $r=0.6$ m, $a=0.24$ m, $x/W=y/W=0.5$, $\theta=0^\circ$

![Figure 4: The right tip stress intensity factors and energy release rate of a crack in a semi-circular anisotropic plate](image)

5. Conclusions

In this paper, the explicit closed-form solution for the stress intensity factors and energy release rates expressed in terms of the remote boundary tractions and displacements have been obtained. The solution is valid for the two-dimensional linear anisotropic elastic problems. Although the closed-form solution has a complex mathematical form, the numerical implementation proves that it is really accurate and efficient. The displacements and tractions of the remote boundaries can be supplied by any method like the finite element method, the experimental measurement, etc.
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References:


