A CUBIC BOUNDARY ELEMENT FOR TWO-DIMENSIONAL COMPOSITE PLATES CONTAINING HOLES

Chyanbin Hwu and T. L. Sun

Institute of Aeronautics and Astronautics
National Chung Kung University
Tainan, 70101, Taiwan, R.O.C

SUMMARY: The Green's function for the two-dimensional composite plates containing an elliptical hole subjected to arbitrary loading around the hole surface is obtained by applying the Stroh's formalism for anisotropic elasticity. By treating this Green's function as a fundamental solution for the boundary integral equation, a boundary element for the loaded-hole problem is developed in this paper. Moreover, to ensure the slope continuity between any two connected element, a cubic element using only the functional values of the nodal points is also developed. Some numerical examples such as cylindrical shell, crack and pin-loaded problems are studied for the purpose of verification and illustration. The results are satisfactory in comparison with other known solutions.

KEYWORDS: boundary element method, composite structures, anisotropy, loaded hole, Green’s function, structural shape optimization

INTRODUCTION

Due to the anisotropy nature of the composite materials, the mechanical behavior of the composite plates is usually studied by using anisotropic elasticity. Under the assumption of two-dimensional linear anisotropic elasticity, a Green’s function satisfying the traction-free-hole boundary conditions has been derived [1]. By treating this Green's function as the fundamental solution for the boundary element method, a linear boundary element was developed [1] and extended to the problems of multi-holes, cracks and inclusions [2]. The special feature of this element is that no meshes are needed around the hole boundary. Thus, a vast of computer time and storage in numerical calculation can be saved. Moreover, due to the exact satisfaction of the hole’s boundary conditions, the results are more accurate than those obtained by using the conventional boundary elements. However, since this is a linear element, the slope between two elements is usually not continuous. Hence, it is not appropriate to apply this element to the works that concern about the slope continuity, such as the structural shape optimization. Moreover, due to the traction-free-hole boundary condition, it is also not appropriate to apply this element to the problems whose holes are subjected to loads.

To improve the element, a new cubic element that is good for the loaded hole and ensures the slope continuity is developed in this paper. By applying the Stroh's complex variable
formalism [3] for anisotropic elasticity, a Green's function satisfying the loaded-hole boundary condition is derived. The shape function of this cubic element is defined by taking the function value and its derivative at the two extreme nodes as unknowns. However, it is not convenient for the user to describe a boundary by using the nodal derivatives. A transformation is therefore introduced to make an equivalent element that contains only the function values of the nodal points.

**FUNDAMENTAL SOLUTION**

Consider an infinite anisotropic plate containing an elliptical hole whose boundary can be expressed by

\[ x_1 = a \cos \psi, \quad x_2 = b \sin \psi, \quad (1) \]

where \( a \) and \( b \) are the lengths of the semi-axis of the ellipse. The plate is subjected to a concentrated force \( \mathbf{p} \) applied at point \( \mathbf{x}^* = (x_1^*, x_2^*) \). In addition, there is a distributed load \( \mathbf{t} \) applied on the hole boundary, which can be expressed by Fourier expansion as

\[ \hat{\mathbf{t}} = \mathbf{c}_0 + \sum_{k=1}^\infty \mathbf{c}_k \cos k\psi + \mathbf{d}_k \sin k\psi, \quad (2a) \]

where

\[ \mathbf{c}_0 = \frac{1}{2\pi} \int_\pi^\pi \hat{\mathbf{t}} d\psi \]

\[ \mathbf{c}_k = \frac{1}{\pi} \int_{-\pi}^\pi \hat{\mathbf{t}} \cos k\psi d\psi \]

\[ \mathbf{d}_k = \frac{1}{\pi} \int_{-\pi}^\pi \hat{\mathbf{t}} \sin k\psi d\psi \quad (2b) \]

and

\[ \rho^2 = a^2 \sin^2 \psi + b^2 \cos^2 \psi \quad (2c) \]

By applying Stroh's formalism for anisotropic elasticity [3], an analytical solution satisfying these loading conditions and all the basic equations such as the strain-displacement equations, the stress-strain laws and the equations of equilibrium, has been found as

\[ \mathbf{u} = 2 \Re \{ \mathbf{Af}(z) \}, \quad \phi = 2 \Re \{ \mathbf{Bf}(z) \}, \quad (3a) \]

where

\[ \mathbf{f}(z) = \mathbf{f}_0(z) + \frac{1}{2\pi i} \log \zeta_a >> \mathbf{A}^T \mathbf{c}_0 - \sum_{k=1}^\infty \frac{1}{2k} \log \zeta_{-k} >> \mathbf{B}^{-1} (i\mathbf{c}_k - \mathbf{d}_k), \quad (3b) \]

and

\[ \mathbf{f}_0(z) = \frac{1}{2\pi i} \log (\zeta_a - \zeta_a^*) >> \mathbf{A}^T \hat{\mathbf{p}} + \sum_{k=1}^\infty \frac{1}{2\pi i} \log (\zeta_{-k} - \zeta_{-k}^*) >> \mathbf{B}^{-1} \mathbf{B} \mathbf{I}_k \mathbf{A}^T \hat{\mathbf{p}} \quad (3c) \]

\[ \zeta_a = \frac{z_a + \sqrt{z_a^2 - a^2 - p_a b^2}}{a - ip_a b} \quad (3d) \]

In the above, \( \mathbf{u} \) and \( \phi \) denote, respectively, the displacement vector and stress function vector which is related to the stresses \( \sigma_0 \) by
\[ \sigma_{i1} = -\phi_{i,2}, \sigma_{i2} = \phi_{i,1}. \]  

(3d)

**CUBIC BOUNDARY ELEMENT**

If body forces are omitted, the boundary integral equation for the boundary value problem in solid mechanics can be written as [4]

\[
c_{ij}(\hat{x})u_j(\hat{x}) + \int_{\Gamma} \hat{T}_j(\hat{x}, x)u_j(x)d\Gamma(x) = \int_{\Gamma} \hat{U}_j(\hat{x}, x)t_j(x)d\Gamma(x)
\]

(4)

where \( \Gamma \) denotes the boundary of the elastic solid. \( u_j(x) \) and \( t_j(x) \) are the displacements and surface tractions along the boundaries. \( c_{ij}(\hat{x}) \) is a coefficient dependent on the location of \( \hat{x} \). For a smooth boundary \( c_{ij} = \delta_{ij}/2 \), in which \( \delta_{ij} \) is the Kronecker delta. For an internal point, \( c_{ij} = \delta_{ij} \). For a practical application, \( c_{ij} \) can be computed by assuming a unit rigid body movement in any one direction. \( \hat{U}_j(\hat{x}, x) \) and \( \hat{T}_j(\hat{x}, x) \) are, respectively, the displacements and tractions in the \( j \) direction at point \( x = (x_1, x_2) \) corresponding to a unit point force acting in the \( i \) direction applied at point \( \hat{x} = (\hat{x}_1, \hat{x}_2) \). That is, \( \hat{U}_j(\hat{x}, x) \) and \( \hat{T}_j(\hat{x}, x) \) are the displacements and tractions associated with the fundamental solution obtained in the previous section.

For the discretization of Eqn (4), the boundary \( \Gamma \) is approximated by using a series of elements. Usually, the Cartesian coordinates \( x \) of points located within each element \( \Gamma_m \) are expressed in terms of interpolation functions \( \omega \) and the nodal coordinates \( x_m \) of the element by the matrix relation \( x = \omega^T x_m \). In a similar way, boundary displacements \( u \) and tractions \( t \) are approximated over each element through interpolation functions \( u = \Psi_u^T u_m \) and \( t = \Psi_t^T t_m \) where \( u_m \) and \( t_m \) contain the nodal displacements and tractions, respectively. Note that the interpolation functions for \( x, u \) and \( t \) may all be different. In this paper, we assume the same variation within each element for the boundary points \( x \), displacements \( u \) and tractions \( t \).

In order to ensure the slope continuity between any two connected elements, the interpolation function of the cubic element is defined by taking the function value and its derivative at the
two extreme nodes as unknowns. Thus, the values of \(x\) (the same applies for \(u\) and \(t\)) at any point on the \(m\)th element can be defined as

\[
x = \phi_1 x_1 + \phi_2 (\partial x / \partial \eta)_1 + \phi_3 x_2 + \phi_4 (\partial x / \partial \eta)_2
\]

and

\[
\phi_1 = (\eta - 1)^2(\eta + 2) / 4 \\
\phi_2 = (\eta - 1)^2(\eta + 1) / 4 \\
\phi_3 = (\eta + 1)^2(2 - \eta) / 4 \\
\phi_4 = (\eta + 1)^2(\eta - 1) / 4
\] (5b)

However, it is not convenient for the user to describe a boundary by using the nodal derivatives. A transformation is therefore introduced to make an equivalent element that contains only the function values of the nodal points. The transformation can be stated as follows: Since only functional values of the nodal points are required to input, the first element is developed by taking the first four nodes of the boundary. Through these four nodal values, a \(4 \times 4\) system of equations can be obtained by using Eqn (5) and the nodal derivatives of the two extreme points of the first element can then be calculated. Because the slope between two connected elements is required to be continuous, the calculated values of the nodal derivatives should be treated as the known values for the next connected element. Thus, only two more nodal points are required to generate the next element, and so on (see Figure 1).

![Figure 1: A finite plate with a loaded hole by using cubic boundary element](image)

After the boundary is discretized into several cubic elements as described previously, following the standard procedure for the boundary element formulation [5] the boundary integral (4) may be set into a system of simultaneous linear algebraic equations. By applying the boundary condition where either the displacement or the traction at each node is prescribed, the system of equations can be solved to obtain all the boundary displacements and tractions. Once all the values of traction and displacement on the boundary are determined, the values of the stresses, strains and displacements at any interior points can also be found through this integral equation [1].

**NUMERICAL RESULTS AND DISCUSSION**

In the following, several interesting examples have been considered to show the accuracy, efficiency and versatility of the present boundary element.
Example 1: A cylindrical shell subjected to external pressure (plane strain solution)

**Without internal pressure**

The main purpose of the present boundary element is the assuredness of the slope continuity. To show this characteristics, a curve boundary is considered in our first example. Consider an isotropic cylindrical shell as shown in Figure 2 with Young’s modulus $E = 11.8$ GPa and poisson’s ratio $\nu = 0.071$ subjected to external pressure $p_a = 1$ GPa acting on the outer surface and $p_b = 0$. The outer and inner radii are 100 mm and 10 mm, respectively. We shall neglect body force, and assume that the distance between any two plane cross sections of the shell normal to the $z$-axis remains constant (assumption of plane strain). A relatively coarse mesh with 8 elements and 16 nodes is employed in this example. From the results shown in Table 1, we observe that the hoop stress around the hole-boundary is independent of the polar angle. This can easily be verified by the inherent axial symmetry of the shell and the applied pressure loading. Moreover, there is a very good agreement between the analytical [5] and computed solution by the present boundary element.

**With internal pressure**

To show that our boundary element is also suitable for the loaded-hole problems, we now add an internal pressure $p_b = 0.5$ GPa to the above problem. The results presented in Table 1 shows that the error of hoop stress on the hole boundary between the analytical solution and present computed solution is 1.32%. Thus, the accuracy of the present method is verified.

![Figure 2: The boundary mesh of cylindrical shell subjected to external pressure and internal pressure](image)

**Table 1: The hoop stress $\sigma_{\theta\theta}$ on the hole-boundary**

<table>
<thead>
<tr>
<th></th>
<th>Present</th>
<th>Exact</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without internal pressure</td>
<td>2.05 GPa</td>
<td>2.02 GPa</td>
<td>1.48%</td>
</tr>
<tr>
<td>With internal pressure</td>
<td>1.53 GPa</td>
<td>1.51 GPa</td>
<td>1.32%</td>
</tr>
</tbody>
</table>

Due to the axial symmetry of the shell, this example can also be solved by the conventional analytical method and our results are therefore compared with the exact solution. The purpose of the numerical methods such as the finite element and boundary element is to solve a complicated boundary value problem of which the analytical solutions are generally untouchable. Therefore, the next example used to verify our element will be a problem whose exact solution does not exist but the solution by using other numerical method can be found in the literature.
Example 2: A rectangular plate with a pin-loaded hole

Here we extract an example that has been done by using a linear boundary element [1]. A rectangular plate with a circular hole is shown in Figure 3. The plate is subjected to a uniform tension applied at $x_2 = -H_2$. A cosine normal load distribution is assumed here to simulate a pin-loaded hole. The pressure around the upper half of the hole boundary induced by a unit stress applied in the $x_2$ direction can be expressed as follow:

$$\sigma_{mm} = P \cos \theta$$

where $\theta$ is measured clockwise from the $x_2$ axis, $m$ is the direction normal to hole boundary, and $P$ is determined by the equilibrium condition that the total forces in the vertical direction balance. The material constants of the orthotropic plate are taken as

$$E_1 = 10 \text{ Gpa}, \quad E_2 = 10 \text{ Gpa}, \quad G_{12} = 3 \text{ Gpa}, \quad \nu_{12} = 0.25$$

Figure 3: A rectangular plate with a pin-loaded hole ($H_1/d = 2.5$, $H_2/d = 7.5$, $W/d = 5$)

In reference [1], the plate is analyzed by superimposing an analytical solution with the results obtained by a linear boundary element that is only suitable for the traction-free hole problems. By the present boundary element, the solution can be obtained directly. Figure 4 shows the results for normal stress $\sigma_{22}$ along the $x_1$ axis, which are almost the same as those presented in [1] and [6].

Figure 4: Normal stress along the $x_1$-axis
Example 3: A Crack in a finite plate

The above two examples consider only circular holes. In order to show that our element is applicable to the general elliptical hole problems. The extreme case of the elliptical hole will be considered in this example. By letting the minor axis $b$ approach to zero, an elliptical hole can be made into a crack of length $2a$. Consider a cracked rectangular plate with width $W = 10$ cm and length $L = 30$ cm as shown in Figure 5. The plate is subjected to a uniform tensile stress $\sigma = 1$ GPa in the $x_2$-direction. The plate is composed of the orthotropic materials whose mechanical properties are

$$E_1 = 114.8 \text{ GPa}, \quad E_2 = E_3 = 11.72 \text{ GPa},$$

$$G_{12} = G_{13} = G_{23} = 9.65 \text{ GPa}, \quad \nu_{12} = \nu_{13} = \nu_{23} = 0.21$$

![Figure 5: A center-crack plate subjected to uniform tension](image)

Due to the square-root singularity of the crack, the comparison of the stresses near the crack tip is meaningless. Instead, the stress intensity factors which play an important role for the understanding the crack problem will be compared in this example. The variation of the stress intensity factor $K_I$ with respect to the crack length $a$ is shown and compared [7] in Figure 6.

![Figure 6: The variation of the stress intensity factor $K_I$ with respect to the crack length $2a$](image)
REFERENCES