Damage and Failure of Interfaces

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The analogy between interface crack problems and punch problems

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ABSTRACT: The interface crack problems and the punch problems are usually solved independently without considering their possible connections. In this paper, we try to re-express the boundary conditions for these two problems and rewrite their corresponding solutions into an analogous form. From the reorganized expressions, we see that if the material above the interface is rigid the solution forms of these two problems can be made to be equivalent by only interchanging the material eigenvectors $A$ and $B$. To get a complete analogous solution, all the boundary conditions including the conditions of the outer boundary should be analogous to each other. A traction-free interface crack problem and a flat-ended punch problem are solved completely in a similar way to illustrate the analogy. From the stress distributions of these two problems, we also observe that the stresses near the interface crack tips and the punch ends possess the same singular order.

1 INTRODUCTION

Both of the interface crack problems and the punch problems belong to the mixed type boundary value problems. The outlook of the boundary conditions of these two problems are very different. The former states the displacement and traction continuity across the uncracked portions, and the traction-prescribed conditions along the cracked portions. The latter (if the punch is considered to be rigid) states the displacement-prescribed condition along the contact regions, and the traction-free condition along the uncontact regions. Because of this difference, these two problems are usually solved independently (Muskheilishvili, 1954; Hwu, 1993a; Fan and Hwu, 1996). However, their solutions show some similarities such as the stress oscillatory singularity characteristics near the interface crack tips or the punch corners. This similarity stimulates us to find the connection between these two problems. By carefully reviewing these two different boundary conditions, we find that the punch problem is just a counterpart of the interface crack problem with one of the materials to be rigid. Hence, similar to the analogy between forces and dislocations, cracks and rigid line inclusions, or holes and rigid inclusion, we may now solve the punch problems by analogy with the interface crack problems, or vice versa. This finding is useful not only in analysis but also in experiment. Because one may understand the physical behavior of the interface crack by doing the experiment of punch problems, or vice versa. This finding is useful not only in analysis but also in experiment. Because one may understand the physical behavior of the interface crack by doing the experiment of punch problems, or vice versa.

2 GOVERNING EQUATION

The basic equations for linear anisotropic elasticity are the strain-displacement equations, the stress-strain laws and the equations of equilibrium, which can be expressed in a fixed rectangular coordinate system $x_i, i = 1, 2, 3$ as (the symbols $x_1$ and $x_2$ will be replaced by $x$ and $y$ for the convenience of presentation)

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

$$\sigma_{ij} = C_{ijkl}e_{kl},$$

$$\sigma_{ij} = C_{ijkl}u_{k,l} = 0,$$

where $u_{i}, \sigma_{ij}$ and $\varepsilon_{ij}$ are respectively the displacement, stress and strain; the repeated indices imply summation; a comma stands for differentiation and $C_{ijkl}$ are the elastic constants which are assumed to be fully symmetric and positive definite.

For two-dimensional problems in which $x_3$ does not appear in the basic equations or the boundary conditions, the general solution to equations (2.1) may be expressed in terms of three holomorphic functions of complex variables (Stroh, 1958; Lekhnitskii, 1963). This enables us to apply many of the powerful results of complex function theory to the two-dimensional elasticity. For the later use of derivation, we now list a compact matrix form solution (Stroh, 1958; Ting, 1986) which satisfies all the basic equations given in (2.1), i.e.,
\[ u = 2 \text{Re}\{A f(z)\} = A f(z) + \overline{A f(z)}, \] (2.2a)
\[ \phi = 2 \text{Re}\{B f(z)\} = B f(z) + \overline{B f(z)}, \] (2.2b)
where
\[ A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}, \]
\[ f(z) = \begin{bmatrix} f_1(z) & f_2(z) & f_3(z) \end{bmatrix}^T, \]
\[ z_a = x + p_a y, \quad a = 1, 2, 3. \]

In the above equations, \( u = (u_1, u_2, u_3) \) is the vector form of displacement; \( \phi = (\phi_1, \phi_2, \phi_3) \) stands for the stress function vector which is related to the stresses \( \sigma_{ij} \) and surface traction \( t \).

3 BOUNDARY CONDITIONS

3.1 Interface crack problems

Consider a set of cracks \( L \) lying along the interface of two dissimilar anisotropic materials. The materials are assumed to be perfectly bonded at all points of the interface except those lying in the region of cracks. To describe the boundary conditions of this kind of interface crack problems, we need to consider the displacement and traction continuity across the uncracked portions, and the traction-prescribed conditions along the cracked portions. Thus,
\[ \phi'_1(x) = \phi'_2(x) = \tilde{t}, \quad x \in L, \]
\[ u_1(x) = u_2(x), \quad \phi_1(x) = \phi_2(x), \quad x \notin L, \] (3.1)
where \( \tilde{t} \) is the prescribed traction along the crack surface; prime (') denotes differentiation with respect to its argument. The symbols marked with the subscripts 1 and 2 represents, respectively, the quantities pertaining to the materials located upper and lower the interface. If we consider the material above the interface to be rigid, the boundary conditions (3.1) will then be specialized to
\[ \phi'_1(x) = \tilde{t}(x), \quad x \in L, \]
\[ u_1(x) = 0, \quad x \notin L. \] (3.2)

Note that in eqn.(3.2) and the following derivation the subscript 2 is dropped for the convenience of presentation, and the subscript 1 will not enter into the boundary conditions since material 1 is assumed to be rigid.

Substituting (2.2a) into (3.2), we have
\[ 2 \text{Re}\{B f'(z)\} = \tilde{t}(x), \quad x \in L, \]
\[ 2 \text{Re}\{A f(x)\} = 0, \quad x \notin L. \] (3.3)

3.2 Punch problems

Consider the case that a set of rigid punches \( L \) of given profiles are brought into contact with the surface of the half-plane and are allowed to indent the surface in such a way that the punches completely adhere to the half-plane on initial contact and during the subsequent indentation no slip occurs and the contact region does not change. The boundary conditions of this kind of punch problems may be expressed by the displacement-prescribed condition along the contact regions, and the traction-free condition along the uncontact regions. Hence,
\[ u_1(x) = \tilde{u}_1(x), \quad x \in L, \]
\[ \phi'_1(x) = 0, \quad x \notin L. \] (3.4)

Substituting (2.2a) into (3.4), we have
\[ 2 \text{Re}\{B f'(x)\} = \tilde{u}_1(x), \quad x \in L, \]
\[ 2 \text{Re}\{A f(x)\} = 0, \quad x \notin L. \] (3.5)

By comparison between eqns.(3.3) and (3.5), we see that they are counterpart of each other. Therefore, we may deal with any one of the problems by analogy with the other problem.

4 THE ANALOGY

One of the special features of the Stroh’s formalism is that the solution form, eqn.(2.2), is neat and elegant. Due to its elegance, many important charac-
characteristics can be found at the first glance of the solution form. For example, the displacements and stress functions shown in eqn.(2.2a) are distinguished only by the material eigenvector matrices \( \mathbf{A} \) and \( \mathbf{B} \). Thus, the relevant boundary conditions of the displacement prescribed problems differ from those of the traction prescribed problems only in the appearance of the symbols \( \mathbf{A} \) and \( \mathbf{B} \). Since the mathematical formulations for the displacement prescribed problems and the traction prescribed problems are identical, their solutions should also be identical with \( \mathbf{A} \) and \( \mathbf{B} \) interchanged.

Both of the interface crack problems and the punch problems belong to the mixed type boundary value problems. Although the outlook of the boundary conditions of these two problems shown in (3.1) and (3.4) are very different, eqn.(3.2) (which is a special case of (3.1)) and eqn.(3.4) are almost identical. In order to see more clearly about their equivalency, we now differentiate the second equation of (3.3) and the first equation of (3.5) with respect to \( x \).

After differentiating, eqns.(3.3) and (3.5) may be rewritten as

interface crack problem: (the material above the interface is rigid)

\[
2 \text{Re} \{ \mathbf{B} \mathbf{J}_o(x) \} = \frac{1}{2 \pi i} \mathbf{X}_o^+(x) \mathbf{p}_n(z), \quad x \in L,
\]

punch problem: (the punch is rigid)

\[
2 \text{Re} \{ \mathbf{B} \mathbf{J}_o(x) \} = \frac{1}{2 \pi i} \mathbf{X}_o^+(x) \mathbf{p}_n(z), \quad x \notin L.
\]

Because \( \mathbf{J}_o(x) \) and \( \mathbf{J}_o(z) \) are given function values in our problem formulation, (4.1) and (4.2) are identical with \( \mathbf{A} \) and \( \mathbf{B} \) interchanged. Therefore, the solutions \( \mathbf{J}(z) \) to these two problems should also be identical with \( \mathbf{A} \) and \( \mathbf{B} \) interchanged. In order to testify our observation, we now list the solutions \( \mathbf{J}(z) \) found in the literature.

Interface crack problem: (the material above the interface is rigid) (Hwu, 1993a)

\[
\mathbf{J}(z) = \mathbf{B}^{-1} \mathbf{M}^{-1} \mathbf{J}_o(z),
\]

\[
\mathbf{J}_o'(z) = \frac{1}{2 \pi i} \mathbf{X}_o^+(x) \int_L \frac{1}{s-x} [\mathbf{x}_o^+(s)]^{-1} \mathbf{J}_o'(s) ds + \mathbf{X}_o^+(x) \mathbf{p}_n(z),
\]

where

\[
\mathbf{X}_o^+(x) = \mathbf{X}_o^-(x), \quad x \notin L,
\]

\[
\mathbf{x}_o^+(x) + \mathbf{M} \mathbf{M}^{-1} \mathbf{x}_o^-(x) = 0, \quad x \in L.
\]

Punch problem: (the punch is rigid) (Fan and Hwu, 1996)

\[
\mathbf{J}(z) = \mathbf{A}^{-1} \mathbf{M}^{-1} \mathbf{J}_o(z),
\]

\[
\mathbf{J}_o'(z) = \frac{1}{2 \pi i} \mathbf{X}_o^+(x) \int_L \frac{1}{s-x} [\mathbf{x}_o^+(s)]^{-1} \mathbf{J}_o'(s) ds + \mathbf{X}_o^+(x) \mathbf{p}_n(z),
\]

where

\[
\mathbf{x}_o^+(x) = \mathbf{x}_o^-(x), \quad x \notin L,
\]

\[
\mathbf{x}_o^+(x) + \mathbf{M}^{-1} \mathbf{M} \mathbf{x}_o^-(x) = 0, \quad x \in L.
\]

In the above equations, \( \mathbf{M} \) is the impedance matrix defined as

\[
\mathbf{M} = -i \mathbf{B} \mathbf{A}^{-1}.
\]

By this definition, it can easily be proved that the interchange of \( \mathbf{A} \) and \( \mathbf{B} \) will lead to the interchange of \( \mathbf{M} \) and \( -\mathbf{M}^{-1} \). \( \mathbf{p}_n(z) \) is an arbitrary polynomial vector with the degree not higher than the number of cracks (or punches) \( n \), which may be determined by the infinity conditions and the single-valuedness requirement of displacements (or equilibrium conditions of each punch): \( \mathbf{x}_o^+(x) \) or \( \mathbf{x}_o^+(x) \) satisfying eqn.(4.3b) or (4.4b) is called the basic Plemelj function matrix whose solution can be found in the Appendix.

Note that the solution form listed in eqns.(4.3) and (4.4) are not exactly the same as that presented in (Hwu, 1993a; Fan and Hwu, 1996). The equivalency between the present solution form and that presented in the literature has been proved in (Hwu and Fan, 1997).

In addition to the analogy shown in (4.1)-(4.4), we also like to present some simple results for the analogy between the surface traction and deformation. It is known that the stress function vector \( \mathbf{\hat{J}} \) and the displacement vector \( \mathbf{u} \) have the following relation (Yeh, et.al., 1993a,b)

\[
\mathbf{A}^T \mathbf{\hat{J}} + \mathbf{B}^T \mathbf{u} = \mathbf{f}(z).
\]

By this relation, many physical quantities can be obtained easily. For the interface crack problems, crack opening displacement \( \mathbf{u} \) along the crack surface \( (x \in L) \) can be found by substituting (4.1) into (4.6), and the stress distribution \( \mathbf{\hat{J}} \) along the in-
terface \((x \not\in L)\) can be found by substituting (4.1) into (4.6). For the punch problems, the contact pressure \(f(x)\) under the punch \((x \in L)\) can be found by substituting (4.2) into (4.6), and the surface deformation \(u(x)\) outside the punch \((x \not\in L)\) can be found by substituting (4.2) into (4.6). The results are:

**interface crack problem:** (the material above the interface is rigid)

\[
y'(x) = (B^T)^{-1}[\dot{f}(x) - A^T \dot{f}(x)], \quad x \in L,
\]

\[
f'(x) = (A^T)^{-1} f(x), \quad x \not\in L; \tag{4.7}
\]

**punch problem:** (the punch is rigid)

\[
y'(x) = (B^T)^{-1}[\dot{f}(x) - B^T \dot{u}(x)], \quad x \in L,
\]

\[
f'(x) = (B^T)^{-1} f(x), \quad x \not\in L. \tag{4.8}
\]

The results obtained in (4.7) and (4.8) show that the surface traction and the displacement gradient are analogous to each other for these two different problems.

5 COMPLETENESS

For a real problem, the boundary conditions given in (4.1) and (4.2) are not complete since they only state the conditions along the interface or the half-plane surface. For a finite body, there should be a condition describing the outer boundary. For an infinite body, the outer boundary condition is the so-called infinity condition. Therefore, without knowing the conditions for the outer boundary (or infinity), the solutions given in (4.3) and (4.4) are incomplete, and the polynomial function vector \(P_n(z)\) remains undetermined. If the outer boundary condition (or infinity condition) does not possess the analogous characteristics like those shown in (4.1) and (4.2), the solutions of \(P_n(z)\) may not have the analogous form for the interface crack problems and the punch problems. In order to have a better understanding about the analogy of \(P_n(z)\), we now choose the examples whose \(\tilde{g}(x)\) (or \(\bar{g}(x)\)) is zero along the crack surface (or the contact region).

5.1 A traction-free interface crack: (the material above the interface is rigid)

Consider a finite interface crack located on \((-a, a)\) subjected to a uniform loading \(\sigma_n^\infty\) at infinity. If the crack surface is traction free, \(\tilde{g}(x) = 0\) and \(\bar{g}(x)\) given in (4.3a) can be simplified to

\[
y'(x) = \bar{X}_o(x) P_1(x), \tag{5.1a}
\]

where

\[
\bar{X}_o(x) = \mathcal{A} \ll \chi o(x) \gg,
\]

\[
P_1(x) = \xi_1 + \zeta_o, \tag{5.1b}
\]

and

\[
\chi o(z) = \frac{1}{\sqrt{z^2 - 2a}} \left\{ \frac{x-a}{z+a} \right\}^{\epsilon o}. \tag{5.1c}
\]

\(\mathcal{A}\) is a matrix satisfying the following eigenrelation

\[
\mathcal{M}^{-1} \mathcal{A} = \mathcal{M}^{-1} \mathcal{A} \ll \epsilon^{2 \epsilon o} \gg. \tag{5.1d}
\]

\(\epsilon o\) is the oscillatory index determined by the material elastic properties (Appendix). \(\xi_1\) and \(\zeta_o\) are coefficient vectors of the polynomial \(P_1(z)\), which will be determined by the infinity condition and the single-valuedness requirement. These two conditions may be expressed as

\[
f = f^{\infty}, \quad \text{when} \quad |x| \to \infty,
\]

\[
\int_{-a}^{a} [y'(x, 0^+) - y'(x, 0^-)] dx = 0, \quad \text{when} \quad |x| \leq a, \tag{5.2a}
\]

where

\[
f^{\infty} = \{\sigma_{12} \sigma_{22} \sigma_{23} \sigma_{33}\}^T. \tag{5.2b}
\]

Applying the results given in (4.7), eqn.(5.2) can be rewritten in terms of \(f(x)\) as

\[
(A^T)^{-1} f'(x) = f^{\infty}, \quad \text{when} \quad |x| \to \infty,
\]

\[
(B^T)^{-1} \int_{-a}^{a} f'(x^-) dx = 0, \quad \text{when} \quad |x| \leq a. \tag{5.3}
\]

Substituting (4.3a), and (5.1) into (5.3) we have

\[
(A^T)^{-1} B^{-1} \mathcal{M}^{-1} \mathcal{A} \ll \chi o(x) \gg \ll \zeta_1 x + \zeta_o \gg = f^{\infty}, \quad \text{when} \quad |x| \to \infty,
\]

\[
\int_{-a}^{a} \chi o(x^-) (\xi_1 x + \zeta_o) dx = 0, \quad |x| \leq a. \tag{5.4}
\]

With the aid of the following integrals (Hwu, 1992)

\[
\int_{-a}^{a} \frac{1}{\sqrt{a^2 - t^2}} \frac{a-t}{a+t} \epsilon o dt = \frac{\pi}{\cosh \pi \epsilon o},
\]

\[
\int_{-a}^{a} \frac{t}{\sqrt{a^2 - t^2}} \frac{a-t}{a+t} \epsilon o dt = \frac{-2i \pi \epsilon o}{\cosh \pi \epsilon o}, \tag{5.5}
\]

eqn.(5.4) now leads to
\[ \zeta_0 \ll 2i\alpha \sigma \gg \zeta_1, \quad \zeta_1 = \Lambda^{-1}B\Lambda^T \zeta_\infty. \]  
\hspace{1cm} (5.6)

Combining the results obtained in (4.3a), (5.1) and (5.6), the complete solution of \( \tilde{f}(z) \) can be written as

\[ \tilde{f}'(z) = B^{-1}M\tilde{f}^{-1}A \ll (z + 2i\alpha \sigma)\chi_0(z) \gg \Lambda^{-1}B\Lambda^T \tilde{f}_\infty. \]  
\hspace{1cm} (5.7)

Substituting (5.7) into (4.7), and using the identity (A 13), we obtain

\[ \tilde{f}'(z) = \frac{-1}{2\pi} A^{-1} \Lambda^{-1} M\Lambda^* \ll \chi_0(z) \gg \Lambda^{-1} q, \]  
\hspace{1cm} (5.13)

which can be proved to be identical to that presented in (Fan and Hwu, 1996). Substituting (5.13) into (4.8), and using the identities (A7), (A11) and (5.8), the contact pressure under the punch can be obtained as

\[ \tilde{f}(z) = \frac{1}{\pi} \Lambda \ll e^{-\pi \sigma} \cosh \sigma \chi_0(x) \gg \Lambda^{-1} q, \]  
\hspace{1cm} (5.15)

5.2 A flat-ended punch: (the punch is rigid)

Consider the indentation by a single flat-ended punch which makes contact with the half-plane over the region \( |x| \leq a \), and the force \( \tilde{q} \) applied on the punch is given. Since the punch end profile is flat, \( \psi'(x) = 0 \) and \( \psi'(z) \) given in (4.4a) can be simplified to

\[ \psi'(z) = \Lambda^* \ll \chi_0(z) \gg (\zeta_1 z + \zeta_0), \]  
\hspace{1cm} (5.10)

where \( \Lambda^* = \frac{B^{-1}M}{\tilde{f}}(4.13a) \) is an eigenvector matrix associated with \( \chi_0(z) \) (Appendix), and the coefficient vectors \( \zeta_0 \) and \( \zeta_1 \) will be determined by the infinity condition and the force equilibrium condition of each punch. These two conditions may be expressed as

\[ t = 0, \quad \text{when } |x| \to \infty, \]  
\hspace{1cm} (5.11)

\[ \int_{-a}^{a} t(x,0^-)dx = \tilde{q}, \quad \text{when } |x| \leq a. \]

Applying the results given in (4.6) and (4.8), eqn. (5.11) can be rewritten in terms of \( \tilde{f}(z) \) as

\[ \tilde{f}'(z) = \tilde{q}, \quad \text{when } |z| \to \infty, \]  
\hspace{1cm} (5.12)

\[ (A^T)^{-1} \int_{-a}^{a} \tilde{f}'(z^-)dx = \tilde{q}, \quad \text{when } |z| \leq a. \]

6 CONCLUDING REMARKS

In this paper, the analogy between the interface crack problems and the punch problems is presented under the consideration that the materials above the interface and the punch are rigid. From the boundary conditions shown in (4.1) and (4.2) along the interface and the half-plane surface, we observe that the solutions to the interface crack problems and the punch problems should have the same forms with \( \chi \) and \( \tilde{f} \) interchanged. This observation is verified in (4.3) and (4.4). In addition to the analogy of the general solutions, the solution forms of the surface traction and the displacement gradient are also analogous to each other, which are shown in (4.7) and (4.8).

It should be emphasized that we state the analogy only by the solution forms not the solution itself. Because for a real problem, the boundary conditions stated in (4.1) or (4.2) are not complete enough. For a finite body, there should be a condition describing the outer boundary. For an infinity body, the outer boundary condition is the so called infinity condition. Therefore, if we want to get an exactly analogous solution, all the boundary conditions should be analogous to each other. The examples presented in Section 5 show that the additional boundary conditions (5.2) or (5.3) for a traction-free interface crack problem, and (5.11) or (5.12) for a flat-ended punch, are not exactly analogous to each other (although (5.3) and (5.12) look alike). Hence, solutions of \( \tilde{f}(z) \)
found in (5.7) for a traction-free interface crack problem and in (5.14) for a flat-ended punch problem cannot be communicated by only interchanging $A$ and $B$.

Although (5.14) cannot be obtained by (5.7) with $A$ and $B$ interchanged, or vice versa, they really reveal the same oscillatory singularity behavior. From the stress distribution shown in (5.9) and (5.15), we observe that the oscillatory singularity characteristics is dominated by the function $X_n(x)$ which is exactly the same for a traction-free interface crack and a flat-ended punch.

The analogy given in this paper is for a rigid punch and a rigid material above the interface. It is hoped that the analogy concept may be extended to the interface crack problems and the elastic contact problems between two dissimilar anisotropic media.

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APPENDIX:

Plemelj function matrix $X_\alpha(z)$ or $X_\alpha^*(z)$

(i) $X_\alpha(z)$

The plemelj function matrix $X_\alpha(z)$ is a sectionally holomorphic function matrix satisfying the following relations

$$X_\alpha^+(x) = X_\alpha^-(x), \quad x \notin L,$$

$$X_\alpha^+(x) + M_xM_x^{-1}X_\alpha^-(x) = 0, \quad x \in L. \quad (A1)$$

The solution to (A1) is (Hwu, 1992)

$$X_\alpha(z) = \Lambda \Gamma(z), \quad (A2a)$$

where

$$\Lambda = \left[ \lambda_1, \lambda_2, \lambda_3 \right],$$

$$\Gamma(z) = \prod_{j=1}^{n} (z - a_j)^{-(1+\delta_\alpha)}(z - b_j)^{\delta_\alpha} \gg . \quad (A2b)$$

$\delta_\alpha$ and $\lambda_\alpha, \alpha = 1, 2, 3$ of (A2b) are the eigenvalues and eigenvectors of

$$\left( e^{2\pi i k} I + M_xM_x^{-1} \right) \lambda_\alpha = 0, \quad (A3a)$$

which can also be expressed as

$$\left( e^{2\pi i k} M_x^{-1} + M_x^{-1} \right) \lambda_\alpha = 0. \quad (A3b)$$

The explicit solutions for the eigenvalues $\delta$ are (Ting, 1986)

$$\delta_\alpha = -1/2 + i\epsilon_\alpha, \quad \alpha = 1, 2, 3, \quad (A4a)$$

where

$$\epsilon_1 = \epsilon = 1/2 \ln \frac{1 + \beta}{1 - \beta}, \quad \epsilon_2 = -\epsilon, \quad \epsilon_3 = 0,$$

$$\beta = -\frac{1}{2} \mathrm{tr}(S^2)^{1/2}, \quad S = i(2AB^T - I). \quad (A4b)$$
$tr$ stands for the trace of matrix. Substituting (A4) into (A3b), we get

$$\tilde{\Lambda}^{-1} = \tilde{\Lambda}^{-1} \ll e^{-2\pi r_a} \gg .$$  \hspace{1cm} (A5)

To have a unique eigenvector matrix, $\tilde{\Lambda}$ is normalized by

$$\tilde{\Lambda}^{-1/2} (M^{-1} + \tilde{M}^{-1}) \tilde{\Lambda} = I. \hspace{1cm} (A6)$$

Combining the normalization equation (A6) with the eigenrelation (A5), we may obtain many useful identities like those presented in (Hwu, 1993b). Shown below is an example which has been used when simplifying the stress distribution vector $\vec{f}(x)$ in (5.9).

$$\left( I + M \tilde{M}^{-1} \right) \tilde{\Lambda} = 2 \tilde{\Lambda} \ll e^{-\pi r_a} \cosh(\pi r_a) \gg .$$  \hspace{1cm} (A7)

(ii) $X^*_+(z)$

The plemej function matrix $X^*_+(z)$ is a sectionally holomorphic function matrix satisfying the following relations

$$X^*_+(z) = X^*_-(z), \quad z \notin L,$$

$$\tilde{M}^{-1} X^*_-(z) = 0, \quad z \in L. \hspace{1cm} (A8)$$

The solution to (A8) is (Hwu, 1992)

$$X^*_+(z) = \tilde{\Lambda}^* I(z), \hspace{1cm} (A9)$$

where $\tilde{\Lambda}^*$ satisfies the following eigenrelation

$$\tilde{M} \tilde{\Lambda}^* = \tilde{M} \tilde{\Lambda}^* \ll e^{-2\pi r_a} \gg .$$  \hspace{1cm} (A10)

From the sectionally holomorphic relations shown in (A1) and (A8), it can be proved that the oscillatory index $r_a, a = 1, 2, 3$, of (A10) are identical to those given in (A4), and

$$\tilde{M} \tilde{\Lambda}^* = \tilde{\Lambda}. \hspace{1cm} (A11)$$

To have a unique eigenvector matrix, $\tilde{\Lambda}^*$ is normalized by

$$\frac{1}{2} \tilde{\Lambda}^* (M + \tilde{M}) \tilde{\Lambda}^* = I. \hspace{1cm} (A12)$$

Combining the normalization equation (A12) with the eigenrelation (A10), we may also obtain many useful identities like those presented in (Hwu, 1993b). Shown below is an example which has been used when deriving the coefficient vector $\vec{L}_o$ in (5.13).

$$\tilde{L}^* \ll e^{\pi r_a} \cosh^{-1}(\pi r_a) \gg = H M \tilde{\Lambda}^*, \hspace{1cm} (A13)$$

where $H$ is a real matrix defined as $H = 2i \tilde{\Lambda} \tilde{\Lambda}^T$ whose inverse equals to $\frac{1}{2}(M + \tilde{M})$. 

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useful tool in solving the pertinent failure cases. The social back­
ground of the historical development of fracture mechanics is high­
lighted and failure cases and incidents of greatest importance
in the history of technology are discussed in detail. The volume is
easy to read and understand as most papers do not require any ma­
thematical background or prerequisite study. Numerous photo­
graphs and drawings illustrate this fascinating and sometimes thrill­
ing development of fracture mechanics.

Azvedo, R.F., E.A. Vargas, L.M. Ribeiro e Sousa & M. Mauns Fer­
nandes (eds.) 90 5410 8649
Applications of computational mechanics in geotechnical engi­
neering – Proceedings of the 2nd international workshop, Rio de
Janeiro, Brazil, 3-5 November 1994
1997, 25 cm, 426 pp., Hfl.195 / $99.00 / €65
Geotechnical works generally present complex engineering pro­lems. The development of constitutive relations for geotechnical
materials, with the help of numerical models, have increased nota­
ibly the ability to predict and to interpret the mechanical behaviour
of these geotechnical works.

Ostinski, Z. (ed.) 90 5410 676 X
Damping of vibrations
1998, 25 cm, 562 pp., Hfl.195 / $100.00 / €65
Sudex. 90 5410 677 8, paper, Hfl.115 / $60.00 / €38
This monograph popularizes and strengthens the considerable
contribution of Polish scientists and engineers to the study of prob­
lems of mechanical vibrations and noise. It presents the latest au­
thor’s research achievements in this domain. The following prob­
lems, among others, are discussed in this book: Structural dam­
ing; Internal damping in composite materials; Damping by dry
friction forces; Vibration damping and stability of rotating systems
supported by slide bearing; Noise attenuation in working machi­

DI Benedetto, H. & L. Francken (eds.) 90 5410 876 2
Mechanical tests for bituminous materials / Essais mécani­
ques pour les matériaux bitumineux – Proceedings of the fifth in­
ternational RILEM symposium / Actes du cinquième symposium
RILEM, Lyon, 14-16 May 1997
1997, 25 cm, 644 pp., Hfl.220 / $110.00 / €73
Bituminous materials are widely used throughout the world mainly for road constructions. A large amount of work and research has been done over the last years and is still continuing, mainly to replace the traditional empirical methods by more fundamental ap­proaches using rational mechanics. The subjects cover binder prop­erties, mixture properties, mix design and field topics. In addition a special chapter is devoted to the presentation of the Rilem Tech­nical Committee interlaboratory tests.

Grebieta, R.H., RAI-Mahaidi & J.L. Wilson (eds.) 90 5410 900 9
Mechanics of structures and materials – Proceedings of the fif­teenth Australasian conference, Melbourne, Victoria, Australia, 8-10 December 1997
1997, 25 cm, 734 pp., Hfl.230 / $115.00 / €77
These proceedings include a selection of a broad range of research topics currently being investigated in the fields of structures, foun­dations and materials in Australasia. Papers cover diverse topics
such as: Fibre reinforced composites; Concrete and steel composi­
tion structures; Concrete design; Steel design and welding; Dy­namics, fire and seismic loading; Finite element analysis and optimi­zation; Stability in structures; Reliability; Timber and masonry
structures; and Geomechanics.

Perez, Jo 90 5410 766 9
Physics and mechanics of amorphous polymers
1998, 24 cm, 224 pp., Hfl.165 / $85.00 / €55
(No rights India)
Contents: Structural aspects of polymers; Molecular mobility in
amorphous solid polymers; Non-elastic deformation of solid
amorphous polymers; Theoretical approach of non-elastic de­
formation of solid amorphous polymers; Mechanical experiments:
Interpretation of results; Physical aging of amorphous polymers;
Glass transition; Conclusion: Should there be one? Appendices.

Lee, P.K.K. (ed.) 90 5410 898 3
Structures in the new millennium – Proceedings of the 4th in­
ternational KereflSicy conference, Hong Kong, 3-5 September 1997
1997, 25 cm, 688 pp., Hfl.250 / $125.00 / €83
Towards the end of the present millenium, developments in struc­
tural engineering have mainly been focussed on ensuring the qual­ity of structures for the service of man and in harmony with the en­vironment. The aim of the conference is to bring together engi­neers and scientists with a common interest in structural engineer­ing to exchange their experiences in recent achievements and to
take a look into the trends of development in the next millenium.
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concept and application; Structural materials – development, ap­plication, durability and environmental impact; Structural types;
buildings, bridges, water and earth retaining structures; Construc­tion of structures - technique, safety, management and repair, etc.

Yuan, Jian-Xin (ed.) 90 5410 904 1
Computer methods and advances in geomechanics – Proceed­ings of the ninth international conference, Wuhan, China, 2-7 No­vember 1997
1997, 25 cm, c.300 pp., 4 vols, Hfl.510 / $255.00 / £170
The proceedings contain 370 papers in addition to keynote lectu­res and invited papers given by the foremost experts in the world.
The contents of this proceedings reflect the latest research and de­velopments as well as applications in geomechanics and geotechn­ical engineering. The technical papers cover major areas of nu­merical and analytical methods, coupling problems, discontinuous
deformation analysis, geoenvironmental engineering, under­ground works, dynamic and cyclic loading, flow and consolid­ation, ground improvement, deep and open pit mining, shallow and
deep foundation, laboratory and in situ testing methods. The vol­umes of these proceedings should serve as an useful tool and invaluable source of intelligence for those engaged in the careers of re­search, education, design, and practical applications in the field of
geomechanics and geotechnical engineering.

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