1. Introduction

Recently, by combining the Stroh formalism for anisotropic elasticity and the method of analytical continuation for the manipulation of a complex variable formulation, we found the analytical solutions for the problem of a dislocation inside, outside or on the interface of an anisotropic elliptical inclusion [1]. It is known that dislocation solutions are frequently used as kernel functions of the singular integral equations which describe the interactions between inclusions and cracks. Based upon the analytical solutions found previously and the numerical technique for solving the singular integral equation, in this paper we consider many kinds of interaction problems like a crack lying inside the inclusion, a crack lying outside the inclusion, a crack penetrating an inclusion or a crack lying along the interface between an inclusion and a matrix. The effects of material properties, crack size and crack location are also studied.

2. Mathematical Formulation

Consider an anisotropic elastic elliptical inclusion imbedded in an infinite matrix, and the dislocation with Burgers vector $\vec{b}$ located at the point $(x_1, x_2)$ which is outside, inside or at the interface of the inclusion. If the inclusion and the matrix are assumed to be perfectly bonded along the interface, the displacements and surface tractions across the interface should be continuous. An elasticity solution satisfying the dislocation singularity and the interface continuity condition has been found in [1]. By representing the cracks as a distribution of dislocations, a singular integral equation of Cauchy type is obtained, which can be solved by a special numerical technique introduced by Gerasoulis [2].

Consider a crack located outside an elliptical anisotropic elastic inclusion subject to uniform loading at infinity (see Figure 1). Due to linearity, the principle of superposition can be used and the problem is represented as the sum of the following two problems: (a) an elliptical anisotropic elastic inclusion embedded in an unbounded anisotropic matrix subject to uniform loading at infinity; (b) an identical problem except that no loading is applied at infinity and the crack surface is subject to the loading which has opposite sense and equal magnitude to...
that obtained from problem (a) at the crack location.

\[ -\frac{1}{2\pi} \int_{s}^{t} L_1 \beta(t) \frac{1}{t-s} \, dt + \int_{s}^{t} \tilde{K}_1(t,s) \beta(t) \, dt = -\tau_n^u(s) \]  

(1a)

where

\[ \tilde{K}_1(t,s) = -\frac{1}{\pi} \text{Re}\{iB_1 \ll \frac{-\gamma}{\zeta_\alpha (\zeta_\alpha - \gamma)} \frac{\partial \zeta_\alpha}{\partial s} \Rightarrow B_1^T\} \]

\[ -\frac{1}{\pi} \sum_{k=1}^{\infty} \text{Re}\{iB_1 \ll -k\zeta_\alpha^{-1} \frac{\partial \zeta_\alpha}{\partial s} \Rightarrow E_k\} \]

(1b)

\[ -\frac{1}{\pi} \sum_{j=1}^{3} \text{Re}\{iB_1 \ll \frac{-1}{\zeta_\alpha (1 - \zeta_\alpha \zeta)} \frac{\partial \zeta_\alpha}{\partial s} \Rightarrow B_1^{-1} B_1 \Rightarrow B_1^{-1}\}, \]

\[ \frac{\partial \zeta_\alpha}{\partial s} = \frac{2\zeta_\alpha^2 (\cos \alpha + p_\alpha \sin \alpha)}{(a - ibp_\alpha)\zeta_\alpha^2 - (a + ibp_\alpha)} \]  

(1c)

\( \tilde{K}_1 \) is a kernel function of the singular integral equation and is H"older-continuous along \(-\ell \leq s \leq \ell\). \( L_1 \) is a real matrix defined as

\[ L_1 = -2iB_1 B_1^T. \]  

(2)

\( \beta(t) \) stands for the dislocation density at point \( t \), which is an unknown function to be determined by the boundary conditions. \( \tau_n^u \) denotes the traction along

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**Figure 1:** The geometry of a crack outside an elliptical inclusion.

As regards problem (a), the solution has been found in [3,4]. For problem (b), we represent the crack as a distribution of dislocation. By integrating the solution for the dislocation located outside the inclusion, and using the superposition principle, a singular integral equation can be formed as [5]
the crack location induced by problem (a). The angular bracket \( \langle \rangle \) stands for the diagonal matrix, i.e., \( \langle f_\alpha \rangle = \text{diag}[f_1, f_2, f_3] \) in which each component is varied according to the Greek index \( \alpha \). The explanation of the other symbols like \( \zeta, \gamma, \beta_1, \beta_2, f_\alpha \), etc. can be found in [1]. If we nondimensionalize the variables \( s \) and \( t \) by letting \( \xi = s/\ell \) and \( \eta = t/\ell \), then eq.(1a) may be rewritten as

\[
\frac{-1}{2\pi} \int_{-1}^{1} L_1 \beta'(\eta) \frac{1}{\eta - \xi} d\eta + \int_{-1}^{1} \tilde{K}_1(\eta, \xi) \beta(\eta) d\eta = -\xi^2(\xi).
\]

(3)

The requirement of crack tip continuity will lead to the following single-valued displacement condition,

\[
\int_{-1}^{1} \beta(\eta) d\eta = 0.
\]

(4)

Since the order of singularity at the crack tip is \(-1/2\), it is convenient to let

\[
\beta(\eta) = \frac{\tilde{\beta}(\eta)}{\sqrt{1 - \eta^2}}
\]

(5)

where \( \tilde{\beta}(\eta) \) is H"older-continuous along \(-\ell \leq s \leq \ell\). Up to now, the entire problem has been reduced to finding the unknown function \( \tilde{\beta}(\eta) \) from the singular integral equations (3)-(5). Through the use of the numerical technique introduced by Gerasoulis [2], the unknown dislocation density \( \beta(t) \) can be determined by the traction-free boundary condition shown in (3), and the single-valued displacement requirement shown in (4).

With the usual definition, the stress intensity factors may now be calculated by [5]

\[
K = \begin{pmatrix}
K_I \\
K_{II} \\
K_{III}
\end{pmatrix} = \frac{\sqrt{\pi \ell}}{2} \tilde{L}_1 \tilde{\beta}(\pm 1) sgn(\pm 1),
\]

(6a)

where \( \tilde{L} \) is the transformation matrix defined as

\[
\tilde{T} = \begin{pmatrix}
-sin \alpha & cos \alpha & 0 \\
-cos \alpha & sin \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

(6b)

and the signum function \( sgn(\xi) \) is defined as \( sgn(\xi) = 1 \) if \( \xi > 0 \) and \( sgn(\xi) = -1 \) if \( \xi < 0 \).

By a similar approach, the singular integral equations for a crack inside an inclusion, a crack penetrating an inclusion, and a curvilinear interface crack lying along the inclusion boundary can all be formulated [5].

3. Numerical Results

Since no analytical solutions exist for the general cases of interactions between cracks and inclusions, comparison should be made with numerical results
obtained by other methods or with analytical results for special cases which can be obtained from the present problems by reduction. The following comparisons have been made by Hwu, et al. [5] for the purpose of verification: interaction between a circular inclusion and an arbitrarily oriented crack (isotropic media) [6], the effect of a rigid elliptical inclusion on a straight crack (isotropic media) [7], interaction between an elastic elliptical inclusion and a crack (anisotropic media) [8], a crack inside a circular inclusion (isotropic media) [9], a crack penetrating a circular inclusion (isotropic media) [9], collinear cracks in homogeneous materials (anisotropic media) [10], a curvilinear crack in homogeneous materials (isotropic media) [11], a curvilinear interface crack between inclusions and matrices (isotropic media) [12].

All the comparisons made above show that our results are correct. With this confidence, we now study the effects of material properties, crack size and crack location on the stress intensity factors. In the following, the material properties of the matrix are chosen to be

\[ E_1 = 114.8 \text{ Gpa}, \quad E_2 = E_3 = 11.72 \text{ Gpa} \]
\[ \nu_{12} = \nu_{13} = \nu_{23} = 0.25 \]
\[ G_{12} = G_{13} = G_{23} = 9.65 \text{ Gpa} \]

while the properties of the inclusion are assumed to be proportional to the matrix as

\[ k = \frac{(E_i)_{2}}{(E_i)_{1}} = \frac{(G_{ij})_{2}}{(G_{ij})_{1}}, \quad i, j = 1, 2, 3, \]

where \( k \) is the index of hardness (or softness). The Poisson ratios \( \nu_{ij} \) of the inclusions are assumed to be identical to those of the matrix. When \( k < 1 \) the inclusion is softer than the matrix, while \( k > 1 \) means hard. A hole or a rigid inclusion can therefore be approximated by letting \( k \to 0 \) or \( k \to \infty \).

![Figure 2: The stress intensity factor for a crack outside an anisotropic elliptical inclusion (\( K_I^* = K_I/\sqrt{\pi l} \sigma_{23}^{23}, b/a = 0.5, l/a = 0.5, \alpha = 0^\circ \)).](image-url)
Figure 3: The stress intensity factor for a crack inside an anisotropic elliptical inclusion ($K^*_I = K_I / \sqrt{\pi l} \sigma_{22}^\infty$, $b/a = 0.5, l/a = 0.25, \alpha = 0^\circ$).

Figure 4a

Figure 4b

Figure 4: The stress intensity factor for a curvilinear interface crack between the matrix and anisotropic circular inclusion ($K^*_I = K_I / \sqrt{\pi a} \sigma_{11}^\infty, K^*_II = K_{II} / \sqrt{\pi a} \sigma_{11}^\infty, \alpha_0 = 0^\circ$).
Figure 2 shows that if the crack is located outside the inclusion, the hard/soft inclusion will diminish/enhance the stress intensity factor of the crack. Moreover, the harder/softer the inclusion is, the lower/higher is the intensity. In addition, the diminution/enhancement effects increase when the crack becomes closer to the inclusion. A similar phenomenon occurs in Figure 3 for a crack inside the inclusion. That is, the harder/softer the matrix is, the lower/higher is the intensity. In other words, the harder/softer the inclusion is, the higher/lower is the intensity. As regards the cases of curvilinear interface cracks between two dissimilar anisotropic media, Figure 4 shows that the harder the inclusion is, the higher is the intensity. However, the stress intensity factors of Mode I and Mode II are not necessary proportional to the square root of the crack length which is in turn proportional to the angle $\alpha$ of the circular segment.

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References

5. C. Hwu, Y.K. Liang, and Yen, W. J., Interactions Between Inclusions and Various Types of Cracks, *submitted for publication*.